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Coherence-Enhancing Diffusion of Colour Images

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Abstract

Many image processing problems require the enhancement of coherent flow-like structures. This can be accomplished in a natural way by combining anisotropic diffusion filtering and texture analysis by means of the structure tensor (second-moment matrix). In this paper an extension of these ideas to vector-valued images is presented. A structure tensor for vector-valued images is constructed as the sum of the structure tensors of each channel. This idea generalizes Di Zenzo's gradient for colour images by introducing an additional integration scale. The common structure tensor is used for steering the diffusion processes in each channel. The advantages of the filter are illustrated by applying it to an example.

Key Words : Anisotropic diffusion, structure tensor, colour.

1 Introduction

Flow-like structures appear in many image processing problems, for instance the automatic assessment of wood surfaces or fabrics, fingerprint analysis, or scientific image processing in oceanography and meteorology. Most humans consider flow-like structures to be pleasant, harmonic, or at least interesting. Thus, it is not surprising that also artists like van Gogh or Munch have emphasized these features in their paintings.

Some images containing flow-like patterns are of poor quality, such that it becomes necessary to enhance them by closing interrupted lines[1]. Moreover, in many "natural" images, colour plays an important role and it would be desirable to make use of this additional information. Vector-valued image processing material with coherent structures, however, may also arise in meteorology and oceanography, if information is collected at different channels corresponding to different wavelengths.

In the present paper we shall address the problem of enhancing such flow-like structures in vector-valued images. To this end, we first have to *analyse* their coherence. This is done by generalizing a well-known tool from texture analysis, the structure tensor (second-moment matrix) to vector-valued images. Its eigenvectors and eigenvalues provide us with all required information. This coherence descriptor enables us to construct a diffusion tensor which steers the diffusion process in each channel in such a way that diffusion is encouraged along the preferred structure orientation.

[1] This problem can even be related to finding illusory contours.

The paper is organized as follows. Section 2 gives a review of the structure tensor concept and it presents its generalization to vector-valued images. Then we discuss in Section 3 how this information is used for designing an appropriate diffusion process. Its properties are illustrated by an example. We conclude with a summary in Section 4.

Related work. The work presented here is a synthesis from two earlier ideas of the author: Using a common structure tensor for vector-valued images was proposed in [10], while scalar-valued coherence-enhancing anisotropic diffusion goes back to [11]. This concept also generalizes work of Whitaker and Gerig [13] for isotropic diffusion of vector-valued images with a common scalar diffusivity to the anisotropic case with a common diffusion tensor. Recently, Chambolle [2] and Sapiro and Ringach [9, 8] have proposed extensions of related image processing equations to colour images. Although revealing several structural similarities to our approach, their intention is edge-preserving smoothing, while the method presented here here achieves coherence-enhancement by looking at an additional, much larger scale: the integration scale. This is also the main difference to the anisotropic diffusion approach of Cottet and Germain [3] for scalar images. Related anisotropic techniques which convolve with a Gaussian, whose shape is adapted to a second moment matrix, go back to Nitzberg and Shiota [5]. Unlike our approach they do not utilize the difference of the eigenvalues of the structure tensor as a coherence measure, and they focus on scalar images. In this sense they are more similar to the methods presented in [12].

2 Analysing coherent structures

For analysing coherent flow-like structures, we first focus on scalar images. Consider a rectangular image domain $\Omega := (0, a_1) \times (0, a_2)$, and let an image $u(x)$ be represented by a bounded mapping $u : \Omega \to \mathbb{R}$.

A very simple structure descriptor is given by ∇u_σ, the gradient of a Gaussian-smoothed version of u:

$$K_\sigma(x) := \frac{1}{2\pi\sigma^2} \cdot \exp\left(-\frac{|x|^2}{2\sigma^2}\right), \tag{1}$$

$$u_\sigma(x,t) := \left(K_\sigma * u(.,t)\right)(x) \qquad (\sigma > 0). \tag{2}$$

The standard deviation σ denotes the *noise scale*, since it makes the edge detector ignorant of details smaller than $O(\sigma)$. Although ∇u_σ is useful for detecting edges, it is unsuited for finding parallel structures, as we can see from Figure 1. The left image shows an original fingerprint. Figure 1(b),(c) depict the gradient orientation using colours: vertical gradients are depicted in red, horizontal ones in green etc. We observe that for small σ high fluctuations remain, while larger σ lead to entirely useless results. This is due to the fact that for larger σ neighbouring gradients with same orientation, but opposite sign cancel each other. Gaussian smoothing averages directions instead of orientations[2]. To make the structure descriptor invariant under sign changes, we may replace ∇u_σ by its tensor product

$$J_0(\nabla u_\sigma) := \nabla u_\sigma \otimes \nabla u_\sigma := \nabla u_\sigma \nabla u_\sigma^T. \tag{3}$$

This matrix is symmetric and positive semidefinite, and its eigenvectors are parallel and orthogonal to ∇u_σ, respectively. The corresponding eigenvalues $|\nabla u_\sigma|^2$ and 0 describe just the contrast in the eigendirections. Now that we have replaced directions by orientations, we can average the orientations by applying a componentwise convolution with a Gaussian K_ρ:

$$J_\rho(\nabla u_\sigma) := K_\rho * (\nabla u_\sigma \otimes \nabla u_\sigma) \qquad (\rho \geq 0). \tag{4}$$

[2]In our terminology, gradients with opposite sign share the same orientation, but point in opposite directions.

This result [7], named *structure tensor* or *second-moment matrix*, has been discovered and used in many different contexts, see for instance the references in [11] for an overview.

It is not hard to verify that the symmetric matrix $J_\rho = \begin{pmatrix} j_{11} & j_{12} \\ j_{12} & j_{22} \end{pmatrix}$ is positive semidefinite and possesses the eigenvalues

$$\mu_{1,2} \;=\; \frac{1}{2}\left(j_{11}+j_{22} \pm \sqrt{(j_{11}-j_{22})^2 + 4j_{12}^2}\right), \tag{5}$$

with $\mu_1 \geq \mu_2$. The corresponding orthonormal set of eigenvectors $\{w_1, w_2\}$ is determined by

$$w_1 \parallel \begin{pmatrix} 2j_{12} \\ j_{22} - j_{11} + \sqrt{(j_{11}-j_{22})^2 + 4j_{12}^2} \end{pmatrix}. \tag{6}$$

The eigenvalues integrate the variation of the grey values within a neighbourhood of size $O(\rho)$. They describe the average contrast in the eigendirections. Thus, the *integration scale* ρ should reflect the characteristic size of the texture. Usually, it is large in comparison to the noise scale σ. The eigenvector w_2 corresponds to the smaller eigenvalue μ_2. It is the orientation with the lowest fluctuations, the so-called *coherence orientation*.

Figure 1(d) depicts this direction. We observe that it is exactly the desired average orientation. It should also be noted how well the singularities correspond to the singularities in the original fingerprint image.

Not only the eigenvectors, but also the eigenvalues provide useful information. Constant areas are characterized by $\mu_1 = \mu_2 = 0$, straight edges give $\mu_1 \gg \mu_2 = 0$, and corners yield $\mu_1 \geq \mu_2 \gg 0$. The expression

$$(\mu_1 - \mu_2)^2 = (j_{11}-j_{22})^2 + 4j_{12}^2 \tag{7}$$

becomes large for anisotropic structures. It measures the *coherence* within a window of scale ρ.

Next we draw our attention to the vector-valued case. We denote a vector-valued image by $\vec{u} : \Omega \to \mathbb{R}^m$ and its channels by u_i, $i = 1,...,m$. Di Zenzo [4] proposed an edge detector for vector images by considering the eigenvalues and eigenvectors of

$$\sum_{i=1}^{m} \nabla u_i (\nabla u_i)^T.$$

If we extend his idea by introducing a noise and integration scale, this comes down to adding the structure tensors of each channel to a common structure tensor

$$J_\rho(\nabla \vec{u}_\sigma) \;:=\; \sum_{i=1}^{m} J_\rho(\nabla u_{i,\sigma}).$$

Its interpretation in terms of eigenvalues and eigenvectors carries immediately over from the scalar case.

3 Coherence-enhancing anisotropic diffusion

Now that we have a tool for *analysing* coherence in vector images, we can try to *enhance* it.

This may be achieved by using anisotropic diffusion filtering. In the scalar case the idea is as follows: One obtains a processed version $u(x,t)$ of an image $f(x)$ with a scale parameter $t \geq 0$ as the solution of a diffusion equation

$$\partial_t u = \mathrm{div}\,(D\;\nabla u), \tag{8}$$

with f as initial condition,

$$u(x, 0) = f(x), \tag{9}$$

and reflecting boundary conditions:

$$\langle D\nabla u,\, n\rangle = 0. \tag{10}$$

Hereby, n denotes the outer normal and $\langle .,. \rangle$ the usual Euclidean scalar product.

The diffusion tensor D is a positive definite 2×2 matrix, which steers the diffusion process: its eigenvalues determine the diffusivities in the directions of the eigenvectors. In the most interesting nonlinear case one adapts the diffusion tensor to the underlying image, for instance to reduce undesired smoothing across edges [12].

The simplest idea to process vector images would be to diffuse each channel separately. If the diffusion tensor depends on the local image structure, however, this causes a risk that a structure (e.g. an edge) evolves at different locations for different channels. Thus, it is plausible to synchronize the evolution by a common diffusion tensor for all channels. In this case the vector-valued diffusion filter has the following structure ($i = 1, ..., m$):

$$\partial_t u_i = \mathrm{div}\,(D\ \nabla u_i), \tag{11}$$

$$u_i(x, 0) = f_i(x), \tag{12}$$

$$\langle D\nabla u_i,\, n\rangle = 0. \tag{13}$$

Since D should take into account information from all channels, a natural choice would be to make it a function of $J_\rho(\nabla \vec{u}_\sigma)$, the structure tensor for vector images. How should this function look like? For enhancing coherence in vector images, we need a smoothing process which acts mainly along the coherence direction w_2 (with the notations from Section 2) and the smoothing should increase with the coherence $(\mu_1 - \mu_2)^2$. This may be achieved in the following way:

We require that D should possess the same eigenvectors w_1, w_2 as the structure tensor $J_\rho(\nabla \vec{u}_\sigma)$. The eigenvalues of D are chosen as

$$\lambda_1 \;:=\; \alpha \tag{14}$$

$$\lambda_2 \;:=\; \begin{cases} \alpha & \text{if } \mu_1 = \mu_2, \\ \alpha + (1-\alpha)\exp\left(\frac{-C}{(\mu_1-\mu_2)^2}\right) & \text{else} \end{cases} \tag{15}$$

with $C > 0$ and a small parameter $\alpha \in (0, 1)$.

We observe that λ_2 is an increasing function in $(\mu_1 - \mu_2)^2$. Since the corresponding eigenvector w_2 is just the coherence direction, we have constructed a diffusion process acting preferably along coherent structures.

The exponential function and the positive parameter $\alpha > 0$ were introduced mainly for two theoretical reasons: It guarantees that the smoothness of the structure tensor carries over to the diffusion tensor. The second reason is that the process never stops: Even if the structure becomes isotropic ($(\mu_1 - \mu_2)^2 \to 0$), we still have isotropic diffusion with diffusivity $\alpha > 0$. Thus, the diffusion tensor is uniformly positive definite.

Exploiting these properties, we may find a well-founded scale-space interpretation in a similar way as for the scalar-valued anisotropic diffusion filters from [10]. This scale-space representation simplifies the vector image with respect to many aspects: within each channel maxima decrease, minima increase, all L^p-norms ($1 \le p \le \infty$) decrease, even central moments are diminished, and the entropy increases. Moreover, the solution depends continuously on the original image. For $t \to \infty$, all channels tend to a constant image. The average value within each channel remains unaltered during the whole evolution. Existence and uniqueness results for this initial boundary value problem can be obtained in a similar way as in [1].

Figure 2(a) gives an example of how coherence-enhancing diffusion works for colour images. It depicts a forest scenery in the winter. The RGB channels were used to regard it as a vector image. In Fig. 2(b) the result after nonlinear isotropic diffusion filtering is shown. It is based on an extension of the methods in [6, 1] for vector images, see [13] for more details. It can be regarded as an edge-preserving smoothing where the (scalar-valued) diffusivity is low whenever $\sum_i |\nabla u_{i,\sigma}|^2$ is large. Figure 2(c) shows the effect of coherence-enhancing anisotropic diffusion. We observe that it gives the desired flow-like simplification of the original image.

4 Conclusions and outlook

We have presented a filter for enhancing coherent structures in vector-valued images. It is based on two ideas: a generalized structure tensor for vector images, and anisotropic nonlinear diffusion filtering with a diffusion tensor. Although we have focused on coherence-enhancement it is evident that the principle also applies to other anisotropic diffusion filters which utilize structure tensor information. For this class of filters we expect interesting applications in the future. It is planned to persue this work presented here by studying the influence of the chosen colour model on the result, and to apply it to other multichannel images arising for instance in medical problems.

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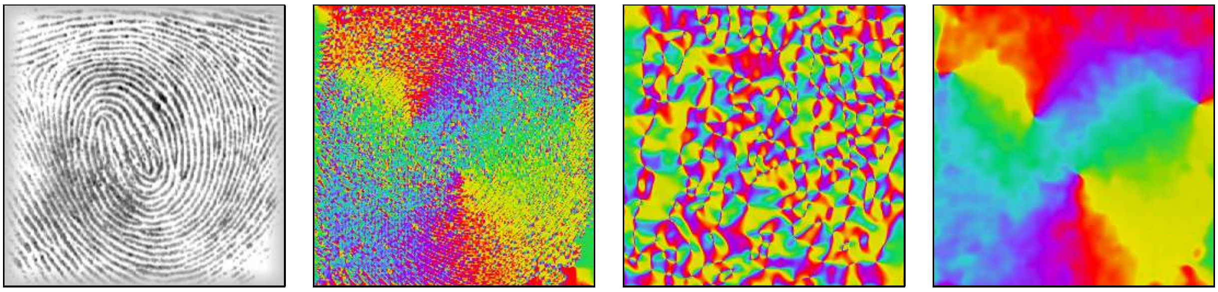

Figure 1: Local orientation in a fingerprint image (revised from [11]). FROM LEFT TO RIGHT: (a) Original fingerprint, $\Omega = (0, 256)^2$. (b) Orientation of smoothed gradient, $\sigma = 0.5$. (c) Orientation of smoothed gradient, $\sigma = 5$. (d) Structure tensor orientation, $\sigma = 0.5$, $\rho = 4$

Figure 2: Nonlinear diffusion of a colour image. (a) LEFT: Original image, $\Omega = (226 \times 323)$. (b) MIDDLE: Isotropic nonlinear diffusion ($\sigma = 2$, $\lambda = 10$, $t = 25$). (c) RIGHT: Coherence-enhancing anisotropic diffusion ($C = 1$, $\sigma = 0.5$, $\rho = 5$, $t = 10$).

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