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CONSERVATIVE IMAGE TRANSFORMATIONS WITH RESTORATION AND SCALE-SPACE PROPERTIES[1]

J.A. Weickert, B.M. ter Haar Romeny, M.A. Viergever

RWCP[1] Novel Function SNN[2] Laboratory
Imaging Center Utrecht, Utrecht University Hospital
E01.334, Heidelberglaan 100, NL–3584 CX Utrecht
E-mail: Joachim.Weickert@cv.ruu.nl

ABSTRACT

Many image processing applications require to solve problems such as denoising with edge enhancement, preprocessing for segmentation, or the completion of interrupted lines. This may be accomplished by applying a suitable nonlinear anisotropic diffusion process to the image. Its diffusion tensor is adapted to the differential structure of the underlying image. Although being image enhancement tools, filters of this class are well-posed. The temporal evolution of the diffusion process creates a scale-space whose causality properties can be understood in a deterministic, stochastic, information theory based and Fourier based way. The well-posedness and scale-space results carry over to the discrete setting, which gives rise to reliable algorithms preserving all properties of the continuous framework.

1. INTRODUCTION

Let an image be represented by a real-valued bounded function f on a rectangular domain $\Omega \in \mathbb{R}^2$. In this paper we are interested in image transformations which reveal the following properties:

(a) We want to simplify f in a gradual way by embedding it into a *scale-space*, i.e. a family $\{T_t f \mid t \geq 0\}$, where $T_0 f = f$, and larger t correspond to "simpler" versions of f. Moreover, we wish to find precise mathematical descriptions to measure this simplicity.

(b) Besides invariances with respect to "natural" image transformations such grey value shifts or image rotations, the transformation T_t should not alter the average grey value of f:

$$\int_\Omega T_t f \, dx = \int_\Omega f \, dx \quad \forall\, t \geq 0.$$

This property is of importance in some scale-space based segmentation algorithms [7].

(c) The class should be flexible in order to incorporate model-based demands such as the enhancement of interesting structures in a stable way.

What is a suitable framework satisfying all these demands ?

Although the scale-space concept offers a large variety a methods [4], scale-spaces can be represented by parabolic partial differential equations with f as initial state [1]. The solution at time $t > 0$ is then regarded as the image representation at scale t.

Conservation of the average grey value can be obtained in a natural way by imposing reflecting boundary conditions and considering processes which can be written in divergence form:

$$\partial_t u = \operatorname{div}(D \, \nabla u).$$

Equations of this type are called *diffusion equations*. In the general *anisotropic* case D is a diffusion tensor, a positive definite 2×2 matrix.

In order to adapt the diffusion process to image restoration tasks, we may choose D as a (nonlinear) function of the underlying local image structure, which is expressed in terms of Gaussian derivatives.

[1]Real World Computing Partnership
[2]Dutch Foundation for Neural Networks

This paper is organized as follows: Section 2 describes the well-posedness of the general filter structure in more detail and shows how this framework can be applied to image restoration problems such as noise elimination with edge enhancement, preprocessing for segmentation, and the enhancement of flow-like textures. In Section 3 we shall study in which sense these transformations can be regarded as smoothing scale-space transformations, and Section 4 sketches how the well-posedness and scale-space results carry over to the practically relevant discrete setting. We conclude with a summary in Section 5. This paper gives a survey on the basic ideas. The reader who is interested in more details is referred to [11, 12, 13] and related work by other authors [9, 2, 8, 3, 4].

2. IMAGE ENHANCEMENT PROPERTIES

2.1. General well-posedness results

We are investigating image transformations which result from solutions $u(x,t)$ of the initial boundary value problem

$$\begin{aligned} \partial_t u = \operatorname{div}(D\,\nabla u) \qquad &\text{on} \qquad \Omega \times (0,\infty), \\ u(x,0) = f(x) \qquad &\text{on} \qquad \Omega, \\ \langle D\,\nabla u,\, n\rangle = 0 \qquad &\text{on} \qquad \partial\Omega \times (0,\infty), \end{aligned}$$

where n is the outer normal vector, $\langle .,. \rangle$ is the Euclidean scalar product, and D is a smooth function of Gaussian derivatives of u, which remains uniformly positive definite (i.e. we have a certain minimal diffusion).

Using methods from [2] it follows that problems of this type have a unique solution which is infinitely often differentiable. Moreover, this solution depends continuously on the original image with respect to the $L^2(\Omega)$ norm [11]. This is of importance when considering stereo images, image sequences or slices from medical CT or MR sequences, since we know that similar images remain similar after filtering.

Our filter class covers linear diffusion filters [6, 14] as well as certian nonlinear isotropic models [2]. However, it also comprises other models which are useful for image enhancement and which we shall discuss next.

2.2. Edge-enhancing anisotropic diffusion

In certain applications one wants to smooth mainly within each region and aims to inhibit diffusion across edges. However, smoothing along edges may be desirable in order to eliminate noise. These demands can be accomplished by choosing the eigenvectors and eigenvalues of D in the following way [11]: We identify edges by means of $\nabla u_\sigma := \nabla(K_\sigma * u)$, where $\nabla u_\sigma := \nabla K_\sigma * u$, K_σ denotes a Gaussian with standard deviation σ, and $*$ is the convolution product. The regularizing effect of using Gaussian derivatives makes this filter insensitive to noise below a scale σ. To adapt the eigenvectors v_1, v_2 of D to ∇u_σ, we require $v_1 \perp \nabla u_\sigma$ and $v_2 \parallel \nabla u_\sigma$. For the corresponding eigenvalues λ_1, λ_2 of D we choose ($C > 0$, $m \in \mathbb{N}$):

$$\begin{aligned} \lambda_1 \;&:=\; \begin{cases} 1 & \text{if } |\nabla u_\sigma| = 0, \\ 1 - \exp\left(\frac{-C}{|\nabla u_\sigma|^{2m}}\right) & \text{else,} \end{cases} \\ \lambda_2 \;&:=\; 1. \end{aligned}$$

Since λ_1 is rapidly decreasing at edges, smoothing on both sides of the edge is much stronger than at the edge itself. As a result, filters of this type may act contrast-enhancing at edges. Indeed, they can be regarded as regularizations of the Perona–Malik filter, which is well-known to reveal such forward–backward diffusion behaviour [9].

Fig. 1 gives an impression of the behaviour of edge-enhancing anisotropic diffusion, when being applied to a MR image of the human head. We recognize that subsequently more and more structures are removed. Due to the significantly reduced diffusivity λ_1, edges remain stable and well-localized across a wide spectrum of scales. The filtered images become piecewise almost constant. This makes subsequent segmentation steps very easy. Examples demonstrating the good noise eliminating and edge-enhancing qualities of this filter class can be found in [11].

2.3. Coherence-enhancing anisotropic diffusion

If one wants to design diffusion filters which enhance coherent flow-like structures, one needs more sophisticated structure descriptors than ∇u_σ. For instance we may use the *structure tensor (second-moment matrix, scatter matrix)* [10]

$$J_\rho(\nabla u_\sigma) := K_\rho * (\nabla u_\sigma \nabla u_\sigma^T).$$

The componentwise convolution with the Gaussian K_ρ averages orientation information over an integration scale ρ. Since J_ρ is a symmetric positive semidefinite matrix, there exists an orthonormal basis of eigenvectors v_1 and v_2 with corresponding eigenvalues $\mu_1 \geq \mu_2 \geq 0$. The eigenvalues measure the average contrast (grey value variation) in the eigendirections within a scale ρ. Therefore, v_1 is the orientation with the highest grey value fluctuations, and v_2 gives the preferred local orientation, the *coherence direction.* The expression $(\mu_1 - \mu_2)^2$ is a measure of the local coherence.

If one wants to enhance coherent structures, one should smooth preferably along the coherence direction v_2 with a diffusivity λ_2 which increases with respect to the coherence $(\mu_1 - \mu_2)^2$. This can be accomplished by

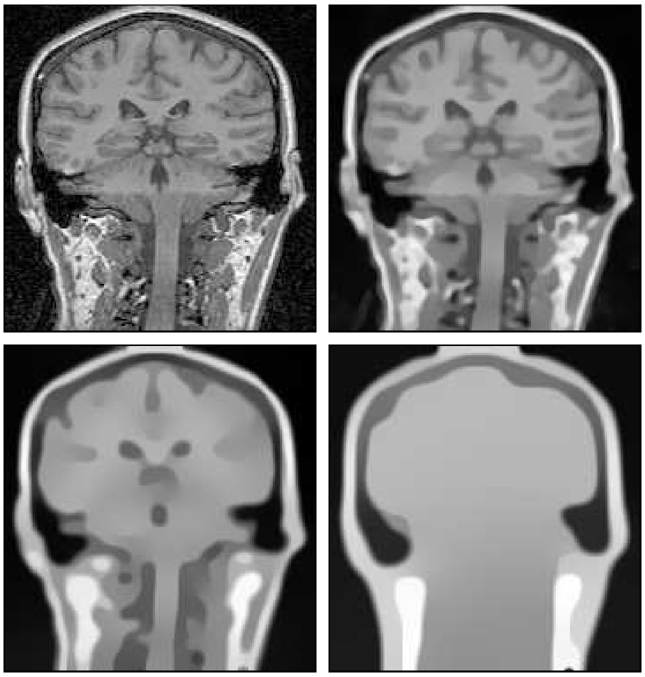

Figure 1: Edge-enhancing anisotropic diffusion filtering of a coronal MR slice of a human head, $\Omega = (0,178) \times (0,187)$, $\sigma = 1$, $C = 850$, $m = 4$. TOP LEFT TO BOTTOM RIGHT: $t = 0$, 10, 80, 640.

designing D such that it possesses the same eigenvectors v_1, v_2 as J_ρ and choosing its corresponding eigenvalues as

$$\begin{aligned} \lambda_1 &:= \alpha, \\ \lambda_2 &:= \begin{cases} \alpha & \text{if } \mu_1 = \mu_2, \\ \alpha + (1-\alpha) \exp\left(\frac{-C}{(\mu_1-\mu_2)^{2m}}\right) & \text{else,} \end{cases} \end{aligned}$$

where, $C > 0$, $m \in \mathbb{N}$, and the small positive parameter $\alpha \in (0,1)$ keeps the diffusion tensor uniformly positive definite.

Fig. 2 shows the restoration properties of coherence-enhancing anisotropic diffusion when being applied to a painting by Vincent van Gogh. We observe that the diffusion filter can close interrupted lines and enhance the flow-like character which is typical for van Gogh paintings. Besides this example, it should be mentioned that coherence-enhancing anisotropic diffusion can also be a useful tool to process fingerprint images and to visualize stripes in fabrics at different scales [12].

3. SCALE-SPACE PROPERTIES

Let us now investigate in which sense we can regard the temporal behaviour of our investigated filters as a scale-space evolution.

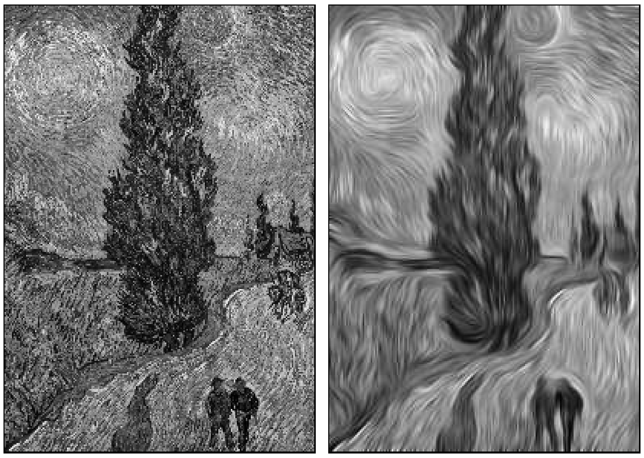

Figure 2: Coherence-enhancing diffusion filtering (a) LEFT: "Lane under Cypresses below the Starry Sky" by van Gogh (Auvers-sur-Oise, 1890; Otterlo, Rijksmuseum Kröller–Müller), $\Omega = (0,203) \times (0,290)$. (b) RIGHT: Filtered, $\sigma = 0.5$, $\rho = 6$, $C = 1$, $m = 4$, $\alpha = 0.001$, $t = 8$.

It is evident that the preceding filters reveal classical architectural scale-space properties such as the semigroup property and typical invariance like invariance under Euclidean transformations or grey level shift invariance. A much more difficult point is the question, in which sense our image-enhancing transformations can be regarded as smoothing, information-reducing filters which do not create new structures.

One important causality requirement is the so-called *evolution property*, which states that no new level-crossings evolve which were not already apparent at finer scales. Hummel [5] has proved that under certain conditions this is equivalent to an extremum principle for linear or nonlinear parabolic operators. Our diffusion filters satisfy the evolution property, since they obey the extremum principle [11]

$$\inf_\Omega f \;\le\; u(x,t) \;\le\; \sup_\Omega f \qquad \forall\, x \in \Omega,\ \forall\, t > 0.$$

Moreover, a large class of useful smoothing properties of nonlinear diffusion filters can be established by considering global measures of simplicity. It can be shown [11] that for all convex, twice differentiable functions r the expression

$$V(t) := \int_\Omega r(u(x,t))\, dx$$

is a *Lyapunov functional:* $V(t)$ is decreasing and bounded from below by $\mathrm{vol}(\Omega)\, r(\mu)$, where μ is the average grey value of f.

On the one hand Lyapunov functionals are of theoretical importance, as they allow to prove that the filtered image converges to a constant image as $t \to \infty$. On the other hand, it follows that – in spite of its image enhancing qualities – our filter class consists of smoothing transformations: Indeed, the special choices $r(s) := |s|^p$, $r(s) := (s-\mu)^{2n}$ and $r(s) := s \ln s$, respectively, imply that all L^p norms with $2 \le p \le \infty$ are decreasing (e.g. the energy $\|u(t)\|^2_{L^2(\Omega)}$), all even central moments are decreasing (e.g. the variance), and the entropy $S[u(t)] := -\int_\Omega u(x,t) \ln(u(x,t))\,dx$, a measure of uncertainty and missing information, is increasing with respect to t [11]. Using Parseval's equality we know that a decreasing energy is equivalent to a decreasing sum of the squared Fourier coefficients. Thus, in spite of the fact that out filters may act image enhancing, their global smoothing properties in terms of Lyapunov functionals can be interpreted in a deterministic, stochastic, information-theory based and Fourier based manner.

4. DISCRETE ASPECTS

From a practical point of view it is of importance that the abovementioned properties carry over to algorithms which work on a discrete pixel grid. In this case it is possible to state reqirements under which one can establish analogous results as in the continuous setting with respect to well-posedness, extremum principles and Lyapunov functionals [13]. It turns out that this formalism also comprises consistent semi-implicit finite difference discretizations of the continuous filters. Since these schemes are absolutely stable, they allow large time step sizes. Moreover, as each step requires only the solution of a sparse linear system of equations, these schemes are fairly efficient.

5. SUMMARY

We have presented a flexible class of well-posed nonlinear anisotropic diffusion filters which combine advantages of image enhancement and scale-space ideas. Lyapunov functionals are the adequate tool to describe them as smoothing scale-space transformations. All well-posedness and scale-space results carry over from the continuous to the discrete setting.

6. REFERENCES

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