Information Measures in Scale-Spaces

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Abstract

This paper investigates Rényi's generalized entropies under linear and nonlinear scale-space evolutions of images. Scale-spaces are useful computer vision concepts for both scale analysis and image restoration. We regard images as densities and prove monotony and smoothness properties for the generalized entropies. The scale-space extended generalized entropies are applied to global scale selection and size estimations. Finally, we introduce an entropy-based fingerprint description for textures.
Keywords: Scale-space, Shannon–Wiener entropy, Rényi's generalized entropies, scale selection, size estimation, texture.

I. Introduction

In recent years multiscale techniques have gained a lot of attention in the image processing community. Typical examples are pyramid and wavelet decompositions. They represent images at a small number of scales and have proven their use for many image processing tasks. Another important class of multiscale techniques consists of so-called *scale-space representations* [1], [2], [3], [4]. They embed an original image into a continuous family of subsequently simpler versions. Many scale-spaces can be formulated as the evolution of the initial image under a suitable linear or nonlinear diffusion process. Such an image evolution is useful for tasks such as feature extraction, scale selection, and segmentation, see [5], [6], [7] and the references therein.

Information theoretical concepts such as the Shannon–Wiener entropy [8], [9], Rényi's generalized entropies [10], [11], [12], and the Kullback–Leibler distance [13] have made contributions to image analysis; for instance Brink and Pendock [14], Brink [15], and Sahoo *et al.* [16] have used them for local image thresholding, and Vehel *et al.* [17] and Chaudhuri and Sarkar [18] study images in a multifractal setting. It is not difficult to see that the generalized entropies, the multifractal spectrum, the gray-value moments and the gray-value histogram itself are equivalent representations: they can be transformed into each other by one-to-one mappings. More details can be found in Appendix A.

Since scale-spaces simplify images, it is only natural to investigate their simplification properties in terms of information measures. Already in 1949, Shannon mentioned that the Shannon–Wiener entropy decreases under averaging transformations [8, p. 52]. In 1993 Illner and Neunzert [19] studied a biased diffusion process, where the original image evolves towards a background image b along a path where its Kullback–Leibler distance with respect to b increases monotonically. Jägersand [20] used the Kullback–Leibler distance in linear scale-space for focus-of-attention. Oomes and Snoeren [21] used the entropy relative to a background measure to estimate the size of objects in images. Sporring [22] applied the Shannon–Wiener entropy in linear scale-space to perform scale selection in textures and showed the monotone behavior using concepts from thermodynamics. Weickert [23] proved monotony of the Shannon–Wiener entropy in linear and nonlinear diffusion scale-spaces by regarding it as a Lyapunov functional. Lyaponov functionals have been used for scale-space synchronization [24] and for a uniform sampling of the scale axis with respect to its information content [25]. The fractal dimension in scale-spaces has been investigated by Peleg *et al.* [26], Barth *et al.* [27], and Pei *et al.* [28]. Relations between Shannon–Wiener entropy and multiscale concepts in terms of wavelets have

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been established by Krim and Brooks [29], where inequality theory was applied to propose optimal measures for feature-directed segmentation.

The present paper extends previous work in this field by studying both theoretical aspects and the practical potential of generalized entropies in a linear and nonlinear multiscale setting. Generalized entropies are complete in the sense that they allow for a reconstruction of the gray-value histogram (see Appendix A). A scale-space extension is used to complement the entropies with spatial information. We prove monotony and smoothness properties with respect to the information order and the scale parameter. We use the scale-space behavior of generalized entropies for scale selection and size estimation, and we introduce a fingerprint-like description for textures. The results indicate that our extensions broaden the potential use of entropy methods in image analysis. Some preliminary results in this paper have been presented at conferences [22], [30].

Throughout this paper we identify an image by its two-dimensional distribution of light on a rectangular image domain. It should be noted that this representation is invariant under multiplication with, but not under addition of, a constant. It is important to note that this two-dimensional distribution is *not* the gray-value histogram.

The outline of our paper is as follows. In Section II will be given a brief introduction to linear and nonlinear scale-spaces. Then in Section III we will investigate a scale-space extension of the generalized entropies. Finally in Section IV we will describe some applications in image processing. A conclusion is given in Section V.

II. A Short Introduction to Scale-Spaces

The images considered in this work are all discrete, but for simplicity we will in this section introduce two scale-spaces in the continuous setting. Discrete scale-space aspects are discussed by Lindeberg [5] for the linear framework, and by Weickert [23] for the nonlinear setting. Scale-spaces can be considered as an alternative to traditional smoothing methods from statistics [31].

In scale-space theory one embeds an image $p(\mathbf{x}) : \mathbb{R}^2 \to \mathbb{R}$ into a continuous family $\{p(\mathbf{x},t) \mid t \geq 0\}$ of gradually smoother versions of it. The original image corresponds to the scale $t = 0$, and increasing the scale should simplify the image without creating spurious structures. Since a scale-space creates a hierarchy of the image features, it constitutes an important step from a pixel-related image description to a semantical image description.

It has been shown that partial differential equations are the suitable framework for scale-spaces [32]. The oldest and best studied scale-space obtains a simplified version $p(\mathbf{x},t)$ of $p(\mathbf{x})$ as the solution of the linear diffusion process with $p(\mathbf{x})$ as initial value.

$$\partial_t p = \partial_{x_1 x_1} p + \partial_{x_2 x_2} p, \tag{1}$$

$$p(\mathbf{x},0) = p(\mathbf{x}), \tag{2}$$

where $\mathbf{x} = (x_1, x_2)^T$. It is well-known from the mathematical literature that the solution $p(\mathbf{x},t)$ can be calculated by convolving $p(\mathbf{x})$ with a Gaussian of standard deviation $\sigma = \sqrt{2t}$:

$$p(\mathbf{x},t) = (G_t * p)(\mathbf{x}), \tag{3}$$

$$G_t(\mathbf{x}) := \frac{1}{4\pi t} e^{\frac{-|\mathbf{x}|^2}{4t}}. \tag{4}$$

This process is called *Gaussian scale-space* or *linear scale-space*. It was first discovered by Iijima [1], [2] and became popular two decades later by the work of Witkin [3] and Koenderink [4]. A detailed treatment of the various aspects of Gaussian scale-space theory can be found in [5], [33], [7] and the references therein.

Unfortunately, Gaussian smoothing also blurs and dislocates semantically important features such as edges. This has triggered people to study nonlinear scale-spaces. Perona and Malik [34] proposed to replace the linear diffusion equation (1) by the nonlinear diffusion process

$$\partial_t p = \nabla \cdot (g(|\nabla p|)\ \nabla p), \tag{5}$$

where $\nabla = (\partial_x, \partial_y)^T$, and the diffusivity $g(|\nabla p|)$ is a decreasing function of $|\nabla p|$. The idea is to regard $|\nabla p|$ as an edge detector and to encourage smoothing within a region over smoothing across boundaries. Thus, locations where the gradient is large have a large likelihood of being an edge, and the diffusivity is reduced.

In our experiments we consider a nonlinear diffusion process where the diffusivity is given by [35]

$$g(|\nabla p|) := \frac{1}{\sqrt{1 + |\nabla p|^2/\lambda^2}} \qquad (\lambda > 0). \tag{6}$$

Such a choice guarantees that the nonlinear diffusion filter is well-posed.

This is one of the simplest representative of nonlinear scale-spaces. Overviews of other nonlinear scale-spaces can be found in [6], [23].

III. Generalized Entropies

Let us now consider a discrete image $\mathbf{p} = (p_1, \ldots, p_N)^T$, where $p_i > 0$ for all i. Note that a single index is used for the two-dimensional enumeration of pixels. Its family of generalized entropies is defined as,

$$S_\alpha(\mathbf{p}) := \frac{1}{1-\alpha} \log \sum_{i=1}^{N} p_i^\alpha \tag{7}$$

for $\alpha \neq 1$. The limit from left and right at $\alpha = 1$ is the Shannon–Wiener entropy,

$$S_1(\mathbf{p}) = -\sum_{i=1}^{N} p_i \log p_i, \tag{8}$$

and we might as well consider it as part of the continuum. The parameter α is called information order.

Let the vector-valued function $\mathbf{p}(t) = (p_1(t), \ldots, p_N(t))^T$ be the linear or nonlinear scale-space extension, where the continuous parameter t denotes scale. These scale-spaces can be obtained by a spatial discretization of Equation 1 or 5 with reflecting boundary conditions.

We will now discuss some details of the mathematical structure of generalized entropies.

Proposition III.1: The generalized entropies are decreasing in α.

Proof: Follows immediately from [10], [36]. ■

Proposition III.2: The generalized entropies $S_\alpha(\mathbf{p}(t))$ are increasing in t for $\alpha > 0$, constant for $\alpha = 0$, and decreasing for $\alpha < 0$. For $t \to \infty$, they converge to S_0.

Proof: The proof is based on a result from [23, Theorem 5]: For a discrete image $\mathbf{p}(t)$, which is obtained from a spatially discrete diffusion scale-space, the following holds. The expression

$$\Phi(\mathbf{p}(t)) := \sum_{i=1}^{N} r(p_i(t)) \tag{9}$$

is decreasing in t for every smooth convex function r. Moreover, $\lim_{t\to\infty} p_i(t) = 1/N$ for all i.

Using this we first prove the monotony of S_α with respect to t. Let $\alpha > 1$ and $s > 0$. Since $r(s) = s^\alpha$ satisfies

$$r''(s) = \alpha(\alpha - 1)s^{\alpha-2} > 0, \tag{10}$$

it follows that r is convex. Thus,

$$\Phi(\mathbf{p}(t)) = \sum_{i=1}^{N} r(p_i(t)) = \sum_{i=1}^{N} p_i^{\alpha}(t) \tag{11}$$

is decreasing in t and

$$S_\alpha(\mathbf{p}(t)) = \frac{1}{1-\alpha} \log \Phi(\mathbf{p}(t)) \tag{12}$$

is increasing in t.
Similar reasonings can be applied to establish monotony for the cases $0 < \alpha < 1$ and $\alpha < 0$.

For $\alpha = 1$ we obtain the Shannon–Wiener entropy for which monotony has already been shown in [23].

Let $\alpha = 0$. Then

$$S_0(\mathbf{p}(t)) = \log \sum_{i=1}^{N} p_i^0(t) = \log N = \text{const.} \quad \forall\, t. \tag{13}$$

To verify the asymptotic behavior of the generalized entropies we utilize $\lim\limits_{t\to\infty} p_i(t) = 1/N$. For $\alpha \neq 1$ this gives

$$\lim_{t\to\infty} S_\alpha(\mathbf{p}(t)) = \frac{1}{1-\alpha} \log \sum_{i=1}^{N} \frac{1}{N^\alpha} = \log N = S_0, \tag{14}$$

and $\alpha = 1$ yields

$$\lim_{t\to\infty} S_1(\mathbf{p}(t)) = -\sum_{i=1}^{N} \frac{1}{N} \log \frac{1}{N} = \log N = S_0. \tag{15}$$

This completes the proof. ■

The following smoothness results constitute the basis for studying derivatives of generalized entropies as will be done in Section IV.

Proposition III.3: The generalized entropies are C^∞ for $\alpha \neq 1$ and at least C^1 in $\alpha = 1$. For linear scale-space they are C^∞ in t, and for the nonlinear scale-space they are C^1 in t.

Proof: In order to prove smoothness with respect to α, we first consider the case $\alpha \neq 1$. Then S_α is the product of the two C^∞ functions $\frac{1}{1-\alpha}$ and $\log \sum_{i=1}^{N} p_i^\alpha$, and thus also C^∞ in α.

The smoothness in $\alpha = 1$ is verified by applying l'Hôpital's rule. Straightforward calculations show that

$$\lim_{\alpha\to 1} \frac{\partial S_\alpha}{\partial \alpha} = \frac{\sum_{i=1}^{N} p_i (\log p_i)^2 - (\sum_{i=1}^{N} p_i \log p_i)^2}{2}. \tag{16}$$

Thus, $\frac{\partial S_\alpha}{\partial \alpha}$ exists and S_α is in C^1.

For linear scale-space, C^∞ in t follows directly from the fact that $G_t(x)$ is in C^∞ with respect to t. In the nonlinear case, C^1 in t is a consequence of the fact that the solution $\mathbf{p}(t)$ is in C^1 with respect to t. This is proven in [23, Theorem 4]. ■

Figure 1 illustrates the monotony of the generalized entropies both in scale and order for both scale-spaces. The figures have been created by finite difference algorithms which preserve the monotony properties established in this section [37].

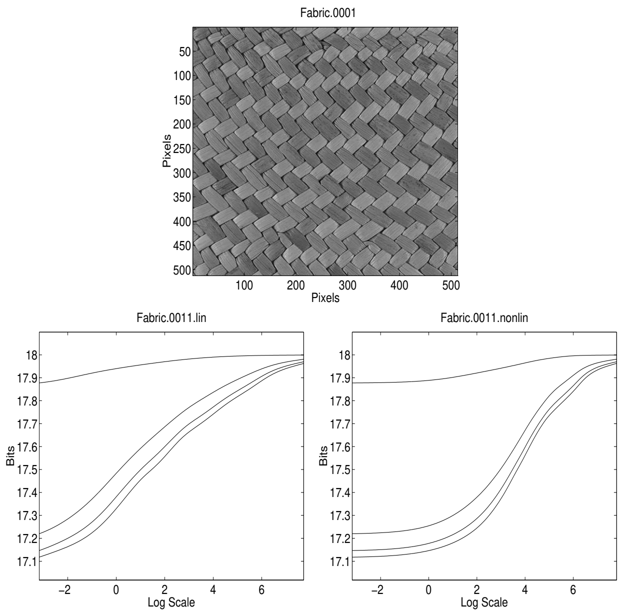

Fig. 1. Examples of some generalized entropies. TOP: A 512×512 gray-valued image. BOTTOM LEFT: Generalized entropies in linear scale-space. From top to bottom $\alpha = 1, 34, 67, 100$. BOTTOM RIGHT: Ditto for nonlinear scale-space.

IV. Experiments

We will in this section demonstrate some applications for the generalized entropies in image processing. We will consider the change of entropies by logarithmic scale,

$$c_\alpha(\mathbf{p}(t)) := \frac{\partial S_\alpha(\mathbf{p}(t))}{\partial(\log t)}, \tag{17}$$

since this appears to be the natural parameter (at least for linear scale-space) [4],[38],[5, section 8.7.1],[30]. We emphasize that the generalized entropies are global measures and are thus best suited for images with homogeneous textures.

A. Shannon–Wiener Entropy and Zooming

This section analyzes the zooming behavior of the Shannon–Wiener entropy in linear scale-space.

Figure 2 (top left and right) shows images from a laboratory experiment. The camera is placed fronto-parallel to a plane with a simple texture: pieces of paper with discs arranged in a regular manner. A sequence is produced as a series of increasing zoom values. In Figure 2 (bottom) we plot the scale $\sigma = \sqrt{2t}$ of the point of maximum entropy change against the mean size of the discs. As can be seen the relation is close to linear. It appears that in linear scale-space the point of maximal entropy change by logarithmic scale corresponds to the size of the dominating image structures.

B. Spatial Extent of Structures

In this section we show that the scaling behavior in linear scale-space carries over to the generalized entropies, and that they can be used to simultaneously measure the size of light and dark structures. We shall also see that the latter cannot be done with the Shannon–Wiener entropy.

The idea is as follows: The definition of the generalized entropies implies that entropies for large positive α focus on high gray-values (white areas), while for large negative value they analyze low gray-values (dark areas).

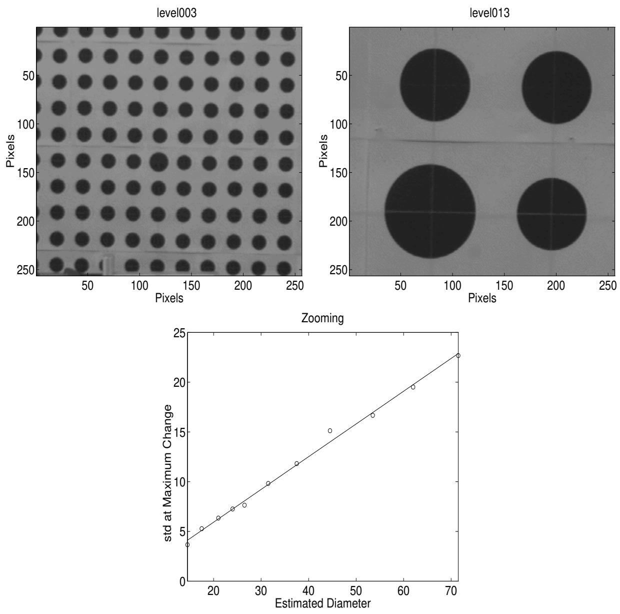

Fig. 2. A zooming sequence. TOP: First and last image. BOTTOM: The $\sigma = \sqrt{t/2}$ values maximizing $c_1(\mathbf{p}(t))$ versus the estimated disc sizes.

We expect that $c_\alpha(\mathbf{p}(t))$ is especially high for structures of diameter d, when the variance $\sigma^2 = t/2$ of the Gaussian is close to the variance of the structures. Let us for simplicity consider a random variable with uniform probability density function whose support is a disc of diameter d. Its variance is,

$$\sigma^2 = \int_0^{2\pi} \int_0^{d/2} \frac{r^2}{\pi (d/2)^2} r \; dr \; d\phi = \frac{d^2}{8}. \tag{18}$$

Hence we expect a light (or dark) structure of diameter d to have a significant entropy change by logarithmic scale at time $\sigma^2/2 = d^2/16$. This size estimate remains qualitatively correct for non-disc structures. In this case, it gives the size of the largest minimal diameter.

Figure 3 shows the result of a performance analysis. The size estimate (18) has been applied to a number of simple sinusoidal images with structures (half wavelengths) between 1 and 257 pixels. As can be seen in the bottom graph, for sufficiently large structures the estimated sizes are close to the true size. Although by definition, the generalized entropies are not symmetric in order, both have a similar scaling behavior which is close to linear.

In Figure 4 we show an experiment on a texture with a more complicated periodicity. This real image has been created by the Belousov–Zhabotinsky reaction [39]. From orders ± 20 we find dominating low intensity values corresponding to a diameter 7.2, while the dominating high intensity values suggest structures of diameter 3.5. From this we conclude that the distance between the light spiral arms in the mean is approximately 7.2 pixels, and the width of the spiral arms is approximately 3.5 pixels. In spite of the fact that the disc model (18) is not very appropriate for the line like structure, the size estimates are in the correct order of magnitude.

The Shannon–Wiener entropy cannot be used for size estimation since it is a mixture of information from both light and dark areas. Thus is does not allow for a distinction between fore- and background.

C. Fingerprints for Entropies in Scale-Space

Section IV-A and IV-B have shown that the scales of extremal entropy change carry significant information for selected information orders. Thus it would be interesting to introduce a compact description of the extremal changes for the continuum of information orders. In analogy with edge

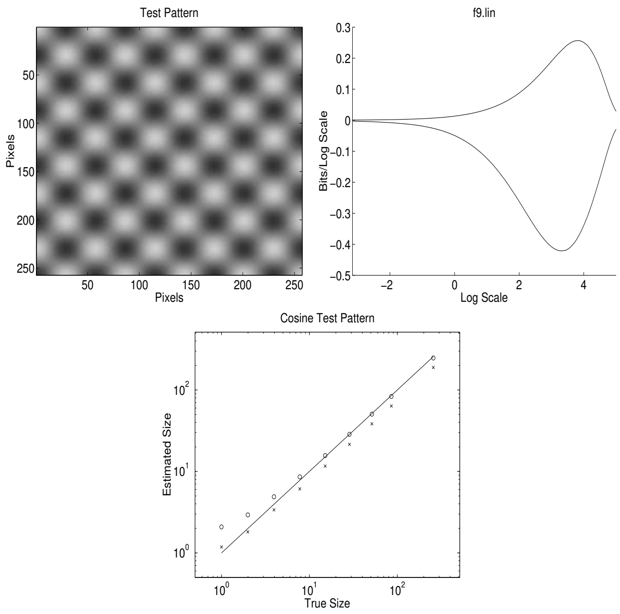

Fig. 3. Scaling behavior and size estimation with generalized entropies. TOP LEFT: Test image generated by $257^{-2}(1+0.6\cos(\omega x_1)\cos(\omega x_2))$ with $\omega = 9\pi/257$. TOP RIGHT: The corresponding $c_\alpha(\mathbf{p}(t))$ curves for $\alpha = \pm 100$. Top curve is for positive order and bottom curve for negative order. BOTTOM: A double logarithmic plot of the true size versus the estimated size for various ω. The straight line depicts the truth, the circles the estimation from order 100, and the crosses for order -100.

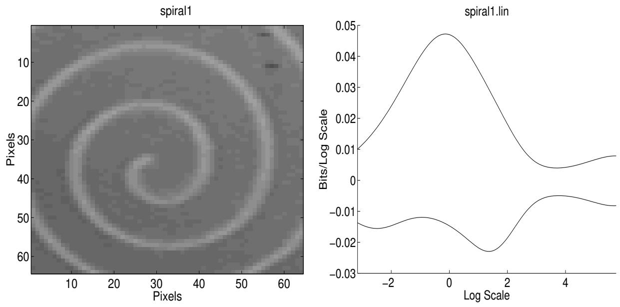

Fig. 4. LEFT: Spiral generated by a chemical reaction. RIGHT: Entropy changes for orders 20 (top curve) and -20 (bottom curve).

analysis in linear scale-space [40] we call such a description a fingerprint image. In Figure 5 fingerprint images for two texture are given, both in the linear and nonlinear scale-space. The fingerprint lines are the extrema of $c_\alpha(\mathbf{p}(t))$ in t. Our monotony results immediately imply the following consequences: If there is only one fingerprint line for a given positive order, then it corresponds to a maximum (likewise to a minimum for negative orders); see also Figure 3. For almost all orders there will be an odd number of fingerprint lines, which correspond to alternating maxima and minima. This can be seen for instance in the middle right graph in Figure 5. For information order 60, the leftmost line is a maximum followed by alternating minima and maxima.

It appears that the location of the fingerprint lines is more stable over information orders for the nonlinear scale-space than for the linear one. Due to the reduced diffusivity of the nonlinear scale-space, the fingerprint lines are shifted towards higher scales.

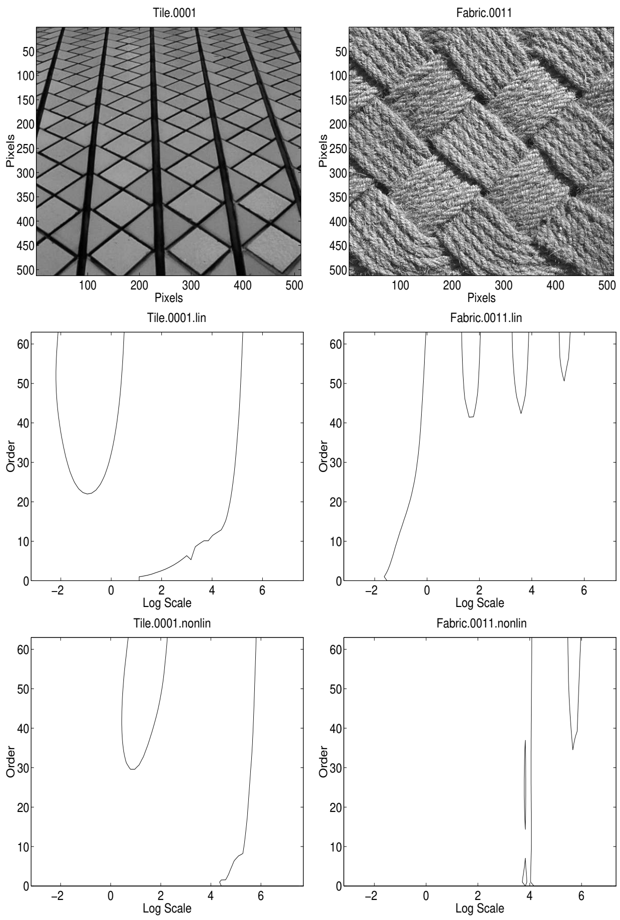

Fig. 5. Fingerprints of generalized entropies. TOP ROW: Two textures. MIDDLE ROW: Fingerprints for linear scale-space. BOTTOM ROW: Ditto for nonlinear scale-space.

V. Conclusions

In this paper we have investigated entropies as a means for extracting information from scale-spaces. This has lead to the following contributions.

- Monotony and smoothness properties for the Shannon–Wiener entropy and Rényi's generalized entropies have been proven for the linear and a nonlinear diffusion scale-space. The proofs hold also for all other nonlinear diffusion scale-spaces treated in [23].
- We have illustrated that the generalized entropies can be used to perform size measurements for periodic textures. This is not possible with the Shannon–Wiener entropy. We have proceeded to define a fingerprint image for entropies in scale-space and analyzed some of its basic properties. The localization of the fingerprint lines can be improved using nonlinear instead of linear scale-space.

The following topics appear promising for future work.

- In the context of texture analysis, it would be interesting to perform an in-depth study on the relation between the fingerprint topology and the structure of the texture.
- This paper has focused on the maximal entropy change by scale to estimate the size of image structures. The minimal change by scale, however, indicates especially stable scales with respect to evolution time. We expect these scales to be good candidates for stopping times in nonlinear diffusion scale-spaces.
- The entropies in this paper are global measures. For topics such as focus-of-attention it would be interesting to study local variants of them.

It should be emphasized that the analysis carried out in this paper is directly transferable to the analysis of multifractals, gray-value moments, and gray-value histograms.

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Appendix

I. Relations to Gray-Value Moments, Histograms, and Multifractal Spectra

The gray-value moments of an image are defined as,

$$m_\alpha(\mathbf{p}) = \sum_{i=1}^{N} p_i^\alpha. \tag{19}$$

From the definition of S_α in (7) it is clear that there is a one-to-one relation to m_α.

Let the image $(p_1, \ldots, p_N)^T$ consist of M distinct gray-values $v_1, \ldots, v_M$ occurring $f_1, \ldots, f_M$ times. We may use this gray-value histogram $\mathbf{f}$ to rewrite the moments as,

$$m_\alpha(\mathbf{p}) := \sum_{j=1}^{M} f_j v_j^\alpha. \tag{20}$$

Considering the moments $m_0, \ldots, m_{M-1}$ gives the relation:

$$\begin{bmatrix} m_0 \\ m_1 \\ m_2 \\ \vdots \\ m_{M-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ v_1 & v_2 & \ldots & v_M \\ v_1^2 & v_2^2 & \ldots & v_M^2 \\ & & \ddots & \\ v_1^{M-1} & v_2^{M-1} & \ldots & v_M^{M-1} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_M \end{bmatrix}$$

The system matrix is a so-called Vandermonde matrix. By induction over M the determinant can be shown to be $\prod_{1 \leq n < m \leq M} (v_m - v_n)$. Since $v_j, j = 1, \ldots, M$ are distinct, the matrix is invertible (but ill-conditioned). Thus there is a one-to-one relation between the moments $m_0, \ldots, m_{M-1}$ and the histogram $f_1, \ldots, f_M$.

The equivalence of the multifractal spectrum and the generalized entropies is discussed in [42].

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