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A semidiscrete nonlinear scale-space theory and its relation to the Perona–Malik paradox

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Abstract. We discuss a semidiscrete framework for nonlinear diffusion scale-spaces, where the image is sampled on a finite grid and the scale parameter is continuous. This leads to a system of nonlinear ordinary differential equations. We investigate conditions under which one can guarantee well-posedness properties, an extremum principle, average grey level invariance, smoothing Lyapunov functionals, and convergence to a constant steady-state. These properties are in analogy to previously established results for the continuous setting. Interestingly, this semidiscrete framework helps to explain the so-called Perona–Malik paradox: The Perona–Malik equation is a forward–backward diffusion equation which is widely-used in image processing since it combines intraregional smoothing with edge enhancement. Although its continuous formulation is regarded to be ill-posed, it turns out that a spatial discretization is sufficient to create a well-posed semidiscrete diffusion scale-space. We also prove that an explicit 1-D Perona–Malik scheme is monotonicity preserving, which explains that staircasing is essentially the only practically appearing instability.

1 Introduction

Although much effort has been spent in the recent decade to establish a theoretical foundation of certain partial differential equations (PDEs) as scale-spaces, it is almost never taken into account that, in practice, images are sampled on a fixed pixel grid[1]. For nonlinear PDE-based filters, usually straightforward finite difference discretizations are applied in the hope that they reflect the nice properties of the continuous equations. Since scale-spaces cannot perform better than

[1] One exception is Lindeberg's semidiscrete linear diffusion scale-space [14].

their numerical discretizations, however, it would be desirable to have a genuinely discrete nonlinear framework which reflects the discrete nature of digital images. In this paper we discuss a semidiscrete scale-space framework for nonlinear diffusion filtering. It keeps the scale-space idea of having a continuous time parameter, while taking into account the spatial discretization on a fixed pixel grid. It leads to nonlinear systems of coupled ordinary differential equations. Conditions are established under which one can prove existence of a stable unique solution which preserves the average grey level. An interpretation as a smoothing scale-space transformation is introduced which is based on an extremum principle and the existence of a large class of Lyapunov functionals comprising for instance p-norms, even central moments and the entropy. They guarantee that the process is not only simplifying and information-reducing, but also converges to a constant image as the scale parameter t tends to infinity.
This semidiscrete framework gives an answer to one of the central problems related to nonlinear diffusion scale spaces: the surprising practical success of the Perona–Malik (PM) filter in spite of its theoretical doubtfulness (*Perona–Malik paradox* [13]). Recently Kichenassamy [12, 13] has made significant contributions to the understanding of this phenomenon for the *continuous* PM equation. In our paper we contrast these results by applying our *(semi-)discrete* theory for explaining this effect. In particular, we shall see that the PM equation – whose continuous formulation is generally regarded to be ill-posed – leads to a well-posed semidiscrete scale-space. Within its stability range, an explicit (Euler forward) time discretization inherits these semidiscrete well-posedness and scale-space properties to the fully discrete setting. Moreover, we prove that its 1-D variant is monotonicity preserving. Thus, a sigmoid-like edge cannot develop oscillations, and the practically observed instabilities are restricted to staircasing effects.

The paper is organized as follows: Section 2 explains the continuous PM filter and presents an m-dimensional semidiscrete formulation. In Section 3 we discuss a semidiscrete well-posedness and scale-space theory for nonlinear diffusion filters, which we apply in Section 4 for establishing well-posedness of the semidiscrete PM scale-space. Section 5 is devoted to fully discrete results, especially the proof of monotonicity preservation. The paper concludes with a summary in Section 6.

2 The Perona–Malik filter

2.1 Continuous formulation

We consider an m-dimensional rectangular image domain $\Omega = (0, a_1) \times \cdots \times (0, a_m)$ with boundary $\partial\Omega$, and a (grey-value) image which is given by a bounded mapping $f : \Omega \to \mathbb{R}$. In order to avoid the blurring and localization problems of linear diffusion filtering, Perona and Malik proposed a nonlinear diffusion method [16]. Their nonuniform process (which they name anisotropic[2]) reduces the diffusivity at those locations which have a larger likelihood to be edges, since they reveal larger gradients. Perona and Malik obtain a filtered image $u(x,t)$ as solution of a nonlinear diffusion equation with the original image as initial condition and reflecting boundary conditions (∂_n denotes the derivative normal to the image boundary $\partial\Omega$):

$$\partial_t u = \operatorname{div}\left(g(|\nabla u|^2)\,\nabla u\right) \quad \text{on} \quad \Omega \times (0,\infty), \tag{1}$$

$$u(x,0) = f(x) \quad \text{on} \quad \Omega, \tag{2}$$

$$\partial_n u = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty). \tag{3}$$

Among the diffusivities they propose is[3]

$$g(|\nabla u|^2) = \frac{1}{1+|\nabla u|^2/\lambda^2} \qquad (\lambda > 0). \tag{4}$$

The experiments of Perona and Malik were visually impressive [16]: edges remained stable over a very long time. It was demonstrated that edge detection based on this process clearly outperforms the linear Canny edge detector.
However, the PM approach reveals some serious problems: It is not hard to see [16] that the PM equation is of forward parabolic type only for $|\nabla u| \leq \lambda$. Regions with $|\nabla u| > \lambda$ are identified as edges, where it may act like a backward diffusion equation across the edge. The forward–backward diffusion behaviour is explicitly intended in the PM method, since it gives the desirable result of blurring small fluctuations and sharpening edges.
On the other hand, backward diffusion is well-known to be an ill-posed process where the solution (if it exists) is highly sensitive even to the slightest perturbations of the initial data.
The current understanding of the PM process is not complete, but there is very much theoretical and practical evidence that such forward–backward processes are ill-posed as well [18, 6, 10, 15, 4, 2, 7, 3, 17]. As one possibility to understand the behaviour of this process it has been suggested to study regularizing approximations where the regularization parameter tends to zero [17, 8, 9]. In this field, however, conjectures are still dominating over established convergence results.
Recently Kichenassamy [12, 13] proved that the PM filter does not even have weak solutions. He introduced a notion of generalized solutions to the PM process, which are piecewise linear and contain jumps, and he analyzed their moving and merging. The current opinion is that, for these solutions, one should neither expect uniqueness nor stability with respect to the initial image [4, 17, 13].
Interestingly, all practically observed instabilities are less severe than one would expect from theory: The main observed instability in simple implementations is the so-called *staircasing effect*, where a smoothed step edge evolves into piecewise linear segments which are separated by jumps. Contributions to the explanation

[2] In our terminology the PM filter is regarded as an isotropic model, since it uses a scalar-valued diffusivity and not a diffusion tensor. For models with a diffusion tensor, see e.g. [19, 20].
[3] For smoothness reasons we write $g(|\nabla u|^2)$ instead of $g(|\nabla u|)$.

and avoidance of staircasing can be found in [23, 3, 1, 5, 12, 13]. In particular, it should be noted that Kichenassamy's generalized solutions are in accordance with these numerical results.
The staircasing effect, however, is mainly visible for fine spatial discretizations and for slowly varying ramp-like edges. Under practical situations, this is hardly observed, and it is an experimental fact that discretizations of the PM are not very unstable. To find an explanation for this phenomenon, let us now investigate a spatially discretized version of this process.

2.2 Semidiscrete formulation

A discrete m-dimensional image can be regarded as a vector $f \in \mathbb{R}^N$, whose components f_i, $i \in J := \{1, ..., N\}$ display the grey values at the N pixels. Pixel i represents the location x_i. Let h_l denote the grid size in l direction. By u_i and g_i we denote approximations to $u(x_i, t)$ and $g(|\nabla u(x_i, t)|^2)$, respectively. Then, a consistent[4] spatial discretization of the PM equation with reflecting boundary conditions can be written as

$$\frac{du_i}{dt} = \sum_{l=1}^{m} \sum_{j \in \mathcal{N}_l(i)} \frac{g_j + g_i}{2h_l^2} (u_j - u_i), \tag{5}$$

where $\mathcal{N}_l(i)$ consist of the two neighbours of pixel i along the direction l (boundary pixels may have only one neighbour) and

$$g_i := g\left(\frac{1}{2}\sum_{l=1}^{m} \sum_{p,q \in \mathcal{N}_l(i)} \left(\frac{u_p - u_q}{2h_l}\right)^2\right) \tag{6}$$

uses a gradient approximation by central differences. In vector–matrix notation (5) becomes

$$\frac{du}{dt} = \sum_{l=1}^{m} A_l(u)\, u \tag{7}$$

where the matrix $A_l(u) = a_{ijl}(u)$ is given by

$$a_{ijl} := \begin{cases} \frac{g_i+g_j}{2h_l^2} & (j \in \mathcal{N}_l(i)), \\ -\sum\limits_{l=1}^{m} \sum\limits_{k \in \mathcal{N}_l(i)} \frac{g_i+g_k}{2h_l^2} & (j = i), \\ 0 & (\text{else}). \end{cases} \tag{8}$$

We observe that a semidiscrete PM process creates a nonlinear system of ordinary differential equations. Let us now study a general well-posedness and scale-space framework for problems of this type.

[4]The originally in [16] proposed scheme is not consistent, which may cause severe deviations from rotational invariance, as can be seen in [15].

3 A semidiscrete scale-space theory

Recently a continuous scale-space interpretation has been established for regularized variants of the continuous PM process [4, 19, 20]. In addition to invariances such as the preservation of the average grey value, it has been shown that – in spite of its contrast-enhancing potential – these equations create smoothing scale-spaces: the obey a maximum–minimum principle, have a large class of smoothing Lyapunov functionals, and converge to a constant steady state [19].
Of course, it is desirable to find discrete approximations which also reveal these qualities *exactly*. To this end, criteria have been identified under which one can guarantee that a semidiscrete scheme of type

$$\frac{du}{dt} = A(u)\,u, \tag{9}$$

$$u(0) = f, \tag{10}$$

possesses such properties [20, 21]. All one has to check are the following criteria for $A(u) = (a_{ij}(u))$:

(S1) Smoothness: $A \in \mathrm{C}^1(\mathbb{R}^N, \mathbb{R}^{N\times N})$.

(S2) Symmetry: $a_{ij} = a_{ji} \quad \forall\, i,j \in J$.

(S3) Vanishing row sums: $\sum_{j\in J} a_{ij} = 0 \quad \forall\, i \in J$.

(S4) Nonnegative off-diagonals: $a_{ij} \geq 0 \quad \forall\, i \neq j$.

(S5) Irreducibility:
We can connect any two pixels by a path with nonvanishing diffusivities.
Formally: For any $i,j \in J$ there exist $k_0,...,k_r \in J$ with $k_0 = i$ and $k_r = j$ such that $a_{k_p k_{p+1}} \neq 0$ for $p = 0,...,r-1$.

Under these prerequisites the filtering process satisfies the following properties [20, 21]:

(a) *Well-posedness:*
The considered problem has a unique solution for all $t > 0$, which depends continuously on the initial value and the right-hand side of the ODE system.
This is of significant practical importance, since it guarantees stability under parameter variations and under perturbations of the original image. Such a property is desirable when considering stereo images, image sequences or slices from medical CT or MR sequences, since we know that similar images remain similar after filtering. It is remarkable that a process which may be edge-enhancing reveals such well-posedness properties.

(b) *Average grey level invariance:*
The average grey level $\mu := \frac{1}{N}\sum_{j\in J} f_j$ is not affected by the semidiscrete

diffusion filter: $\frac{1}{N}\sum_{j\in J} u_j(t) = \mu$ for all $t > 0$.
This invariance is useful for scale-space based segmentation algorithms and for all applications where grey values are related to physical qualities, for instance in medical imaging.

(c) *Extremum principle:*

$$\min_{j\in J} f_j \leq u_i(t) \leq \max_{j\in J} f_j \qquad \forall\, i \in J, \quad \forall\, t \in [0,\infty).$$

This property is much more than a stability result which forbids under- and overshoots. Since it also ensures that iso-intensity linking towards the original image is possible, it states an important causality property, cf. [11].

(d) *Smoothing Lyapunov functionals:*
In spite of some possible edge enhancement, the considered processes are simplifying, information-reducing transformations with respect to many aspects: The p-norms $\|u(t)\|_p := (\sum_j |u_j(t)|^p)^{1/p}$ are decreasing in t for all $p \geq 2$, all even central moments $M_{2n}[u(t)] := \frac{1}{N}\sum_j (u_j(t)-\mu)^{2n}$ are decreasing in t, and the entropy $S[u(t)] := -\sum_j u_j(t) \ln u_j(t)$, a measure of uncertainty and missing information, is increasing in t (if f_j is positive for all j).

(e) *Convergence to a constant steady state:*
The scale-space evolution tends to the most global image representation that is possible: a constant image with the same average grey level as f:

$$\lim_{t\to\infty} u_i(t) = \mu \quad \forall\, i \in J.$$

These semidiscrete results are part of a more general framework which includes also a continuous and fully discrete theory [21]. Full details with proofs can be found in [20], where it is also shown that it is possible to derive semidiscrete scale-spaces from spatial discretizations of continuous scale-spaces. Due to its ill-posedness, however, the continuous PM equation cannot be treated within this continuous scale-space framework.

4 Application to the semidiscrete Perona-Malik process

Interestingly, it is not hard to verify that the semidiscrete PM process satisfies the well-posedness and scale-space requirements (S1)–(S5).

(S1) Since $g \in C^\infty(\mathbb{R})$, it follows from (8) that $A := \sum_l A_l \in C^\infty(\mathbb{R}^N, \mathbb{R}^{N\times N})$.

(S2) The symmetry of A follows directly from (8) and the symmetry of the neighbourhood relation ($i \in \mathcal{N}_l(j) \iff j \in \mathcal{N}_l(i)$).

(S3) By the construction of A it is also evident that all row sums vanish.

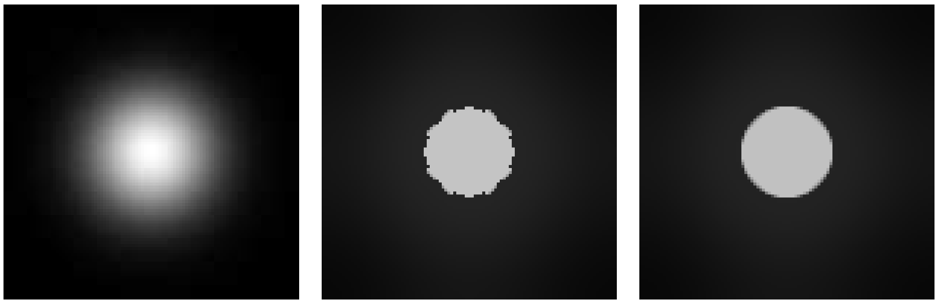

Figure 1: Nonlinear diffusion of a Gaussian-like image. (a) LEFT: Original image, $\Omega = (0,101)^2$. (b) MIDDLE: Processed with the PM filter, $\lambda = 9$, $t = 250$. (c) RIGHT: Regularized PM filter [4], $\lambda = 9$, $\sigma = 0.7$, $t = 250$.

(S4) Since g is positive, it follows that $a_{ij} \geq 0$ for all $i \neq j$.

(S5) In order to show that A is irreducible, let us consider two arbitrary pixels i and j. Then we have to find $k_0,...,k_r \in J$ with $k_0 = s_1$ and $k_r = s_2$ such that $a_{k_q k_{q+1}} \neq 0$ for $q = 0,...,r-1$. If $i = j$, we know already from (8) that $a_{ii} < 0$. In this case we have the simple path $i = k_0 = k_r = j$. For $i \neq j$, we may choose any arbitrary path $k_0,...,k_r$, such that k_q and k_{q+1} are neighbours for $q = 0,...,r-1$. Then,

$$a_{k_q k_{q+1}} \;=\; \frac{g_{k_q} + g_{k_{q+1}}}{2h_l^2} \;>\; 0$$

for some $l \in \{1,...,m\}$. This completes the proof of (S5).

We observe that – despite the fact that the PM filter is regarded to be ill-posed in the continuous setting, its semidiscrete approximation on a fixed grid satisfies (S1)–(S5) and, thus, reveals all the beforementioned well-posedness and scale-space properties. Hence, the spatial discretization causes an implicit regularization[5]. It is caused by the fact that the extremum principle limits the modulus of discrete gradient approximations.
The grid regularization, however, is certainly not the best regularization strategy. Experiments show that grid refinement leads to more staircasing [15, 7, 3]. Moreover, grid regularization may cause artefacts which reflect the grid structure (Fig 1(b)). One can avoid these problems by introducing the regularization explicitly in the continuous PM model, for instance by a Gaussian smoothing of ∇u within the diffusivity [18, 4, 15, 23, 19, 20]. Figure 1(c) illustrates that – as soon as the standard deviation σ of the Gaussian reaches the pixel size – this regularization dominates over the pixel regularization and avoids their artefacts. Moreover, since it bounds the gradient already in the continuous formulation, it leads to continuous models which satisfy similar well-posedness and scale-space

[5] Such a regularizing effect has already been conjectured in [4].

properties as in the semidiscrete case [19, 20]. Last but not least, it makes the filter more robust against noise [4] and reduces staircasing [15, 7, 3]. Thus, such regularizations are generally recommendable.

5 Fully discrete results

We have seen that a spatial discretization is sufficient to make the PM filter well-posed. Of course, in practice, one has to apply a temporal discretization as well. If we consider a time step size $\tau > 0$ and denote by u_i^k an approximation of $u(x_i, k\tau)$, then the simplest fully discrete PM equation is given by the explicit scheme

$$\frac{u_i^{k+1} - u_i^k}{\tau} = \sum_{l=1}^{m} \sum_{j \in \mathcal{N}_l(i)} \frac{g_j^k + g_i^k}{2h_l^2} \, (u_j^k - u_i^k). \tag{11}$$

In [20] it is shown that such a scheme inherits its well-posedness and scale-space properties from the semidiscrete one if the time step size satisfies the stability restriction[6] $\tau < 1/\sum_{l=1}^{m} \frac{2}{h_l^2}$. So we do not have to worry that we loose the nice semidiscrete results by the temporal discretization.
Finally, we prove an important property which is responsible for the fact that – apart from the grid artefacts depicted in Fig. 1 – staircasing is essentially the only practical instability that is observed when implementing the PM equation: the one-dimensional discrete PM process is *monotonicity preserving* [3]: if the grey values are increasing (decreasing) from left to right, then they remain increasing (decreasing) after filtering. Together with the extremum principle, this property explains why over- and undershoots and the creation of other oscillations are not possible.

Proposition 1 *For $\tau < \frac{h_1^2}{2}$, the 1-D explicit PM scheme is strictly monotonicity preserving. For a consecutive pixel numbering this means that*

$$u_i^k < u_{i+1}^k \quad \forall\, i \qquad \Longrightarrow \qquad u_i^{k+1} < u_{i+1}^{k+1} \quad \forall\, i.$$

Proof: Consider some arbitrary inner pixel $i \in \{2, ..., N-2\}$. From

$$\begin{aligned} u_i^{k+1} &= \tau\, a_{i,i+1}\, u_{i+1}^k + (1 - \tau\, a_{i,i+1} - \tau\, a_{i,i-1})\, u_i^k + \tau\, a_{i,i-1}\, u_{i-1}^k \\ u_{i+1}^{k+1} &= \tau\, a_{i+1,i+2}\, u_{i+2}^k + (1 - \tau\, a_{i+1,i+2} - \tau\, a_{i+1,i})\, u_{i+1}^k + \tau\, a_{i+1,i}\, u_i^k \end{aligned}$$

and the symmetry of A we obtain

$$u_{i+1}^{k+1} - u_i^{k+1} = \tau\, a_{i+1,i+2}\, (u_{i+2}^k - u_{i+1}^k) + (1 - 2\tau a_{i,i+1})\, (u_{i+1}^k - u_i^k) + \tau\, a_{i-1,i}\, (u_i^k - u_{i-1}^k).$$

Let $\tau < \frac{h_1^2}{2}$ and $u_j^k < u_{j+1}^k$ for all j. Since $0 < a_{j,j+1} \le 1/h_1^2$ for all j, we know that all summands of the RHS are positive. Thus, it follows that $u_i^{k+1} < u_{i+1}^{k+1}$ for $i \in \{2, ..., N-2\}$. It is easy to see that the same reasoning is applicable to the modified equations arising at the boundaries when $i = 1$ or $N-1$. q.e.d.

[6] Much more efficient absolutely stable schemes are proposed in [22].

6 Conclusions

In this paper we have discussed a genuinely semidiscrete nonlinear theory for nonlinear diffusion scale-spaces. As one example for the importance of such a theory we have applied it in order to explain the well-posedness of the semidiscrete Perona–Malik scale-space. These results also carry over to the fully discrete setting, where a monotonicity preserving property has been proved. They explain that observed instabilities of PM discretizations are restricted to staircasing effects and grid artefacts affecting the rotational invariance. These effects may be tamed by introducing a grid independent spatial regularizations already in the continuous equation [18, 4, 15, 20].

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