Approximation Related to Quotient Functionals

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Abstract

We examine the best approximation of componentwise positive vectors or positive continuous functions f by linear combinations $\hat{f} = \sum_j \alpha_j \varphi_j$ of given vectors or functions φ_j with respect to functionals Q_p, $1 \le p \le \infty$, involving quotients $\max\{f/\hat{f}, \hat{f}/f\}$ rather than differences $|f - \hat{f}|$. We verify the existence of a best approximating function under mild conditions on $\{\varphi_j\}_{j=1}^n$. For discrete data, we compute a best approximating function with respect to Q_p, $p = 1, 2, \infty$ by second order cone programming. Special attention is paid to the Q_∞ functional in both the discrete and the continuous setting. Based on the computation of the subdifferential of our convex functional Q_∞ we give an equivalent characterization of the best approximation by using its extremal set. Then we apply this characterization to prove the uniqueness of the best Q_∞ approximation for Chebyshev sets $\{\varphi_j\}_{j=1}^n$.

1 Introduction

In various applications, e.g., in query optimization [3, 7] or in the restoration of images contaminated with multiplicative noise [13, 2] it is useful to involve quotients rather than differences into the mathematical models and to ask for positive solutions. Moreover, generalized relative error measures [8, 11, 17] make use of quotients.

In this paper, we consider the approximation of positive discrete or continuous functions f by linear combinations $\hat{f} = \sum_{j=1}^n \alpha_j \varphi_j$ such that a certain functional Q_p, $1 \le p \le \infty$, is minimized. The functional Q_p resembles the L_p norm of the function $\max\{\hat{f}/f, f/\hat{f}\} - 1$ for $\hat{f} > 0$. More precisely, we are interested in a minimizer of $Q_p(A\cdot)$, where A denotes the linear transform $A\alpha := \sum_{j=1}^n \alpha_j \varphi_j / f$. A simple example is the approximation of a componentwise positive vector $(f(x_i))_{i=1}^m$ by data $(\hat{f}(x_i))_{i=1}^m$ lying on a line $\hat{f}(x) = \alpha_1 + \alpha_2 x$ with respect to the Q_∞ functional. Then we search for coefficients α_1, α_2 such that

$$\max_{i=1,\ldots,m} \max\left\{\frac{\hat{f}(x_i)}{f(x_i)}, \frac{f(x_i)}{\hat{f}(x_i)}\right\}$$

becomes minimal and $\hat{f}(x_i) > 0$, $i = 1, \ldots, m$. Of course, due to $\ln(\max\{\hat{f}/f, f/\hat{f}\}) = |\ln f - \ln \hat{f}|$ one could minimize $\| \ln f - \hat{f}\|_p$ and use $e^{\hat{f}}$ as approximation of f. However,

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as demonstrated in our numerical Example 3.1 this is often not a good choice.

This paper is organized as follows: In Section 2, we introduce the quotient functionals Q_p, $1 \le p \le \infty$, and verify their convexity and continuity. We prove that under mild conditions on $\{\varphi_j\}_{j=1}^n$ the functional $Q(A\cdot)$ attains a minimum and that the minimizer is unique for $1 < p < \infty$ if A has nullspace $\{0\}$. In Section 3, we deal with discrete data. We compute a minimizer of $Q_p(A\cdot)$, $p = 1, 2, \infty$ by second order cone programming. The best approximation with respect to the Q_∞ functional is examined in Section 4. Once we have computed the subdifferential of Q_∞, the approach follows basically the lines in [14], but with all the necessary modifications due to the fact that Q_∞ is not a norm. We give an equivalent characterization of the minimizer of $Q_\infty(A\cdot)$ using its extremal set and apply this characterization to prove the uniqueness of the minimizer if A is related to a Chebyshev set. We show the relation of our results to the best approximation with respect to a generalized relative error.

2 Quotient functionals

Our considerations are based on the '*quotient function*' $q : \mathbb{R} \to [0, \infty]$ given by

$$q(x) := \begin{cases} x - 1 & \text{for } x \in [1, \infty), \\ \frac{1}{x} - 1 & \text{for } x \in (0, 1), \\ \infty & \text{otherwise}, \end{cases} \tag{1}$$

i.e., $q(x) = \max\{x - 1, \frac{1}{x} - 1\}$ for $x > 0$. The function q is convex and continuous and $\mathrm{dom}(q) := \{x : q(x) < \infty\}$ is open, see also [12, p. 52, 83] and Fig. 1 left.
Let Ω be either a (innumerable) compact subset of $\mathbb{R}^d$ and μ the Lebesgue measure on Ω or a finite subset $\{x_1, \ldots, x_m\}$ of $\mathbb{R}^d$ with point measure μ. By $X := C(\Omega)$ we denote the space of continuous functions on Ω, resp. the space $X := \mathbb{R}^m$ and by $X_{>0}$ the positive functions in X. Set

$$Q(x, f) := q(f(x)) = \begin{cases} f(x) - 1 & \text{for } f(x) \in [1, \infty), \\ \frac{1}{f(x)} - 1 & \text{for } f(x) \in (0, 1), \\ \infty & \text{otherwise}. \end{cases} \tag{2}$$

Proposition 2.1 *The function $Q : \Omega \times X \to [0, \infty]$ in (2) is continuous in x for every $f \in X$ and convex in f for every $x \in \Omega$.*

Proof: The continuity of $Q(\cdot, f)$, $f \in X$, follows by the continuity of f and q and the convexity of $Q(x, \cdot)$, $x \in \Omega$ by the convexity of q. $\square$

We want to concatenate the quotient function with the L_p norms

$$\|f\|_p := \left(\int_\Omega |f(x)|^p \, d\mu \right)^{1/p}, \quad 1 \le p < \infty, \quad \text{and} \quad \|f\|_\infty := \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|.$$

For $1 \le p \le \infty$, we introduce $Q_p : X \to [0, \infty]$ by

$$Q_p(f) := \begin{cases} \|Q(\cdot, f)\|_p & \text{if } \mu\{x : f(x) \le 0\} = 0, \\ \infty & \text{otherwise.} \end{cases}$$

For example, we obtain for $f(x) := \sqrt{x}$ on $\Omega := [0,1]$ that $Q_1(f) = 1$ while $Q_p(f) = \infty$ for $p > 1$. For $p = \infty$ we have that

$$Q_\infty(f) := \sup_{x \in \Omega} Q(x, f).$$

In particular, we see in the case $X = \mathbb{R}^m$ that $Q_p(f) = \left(\sum_{i=1}^m Q(x_i, f)^p\right)^{1/p}$, $1 \le p < \infty$ and $Q_\infty(f) = \max_{i=1,\dots,m} Q(x_i, f)$.
The level sets $\{f \in \mathbb{R}^2 : Q_p(f) \le 1\}$ for $p = 1, 2, 3, \infty$ are illustrated in Fig. 1 right.

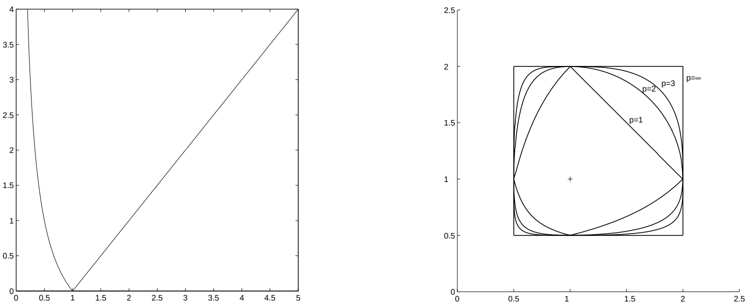

Figure 1: Left: The function q. Right: The 'unit sphere' of Q_p for $p = 1, 2, 3, \infty$ in $\mathbb{R}^2$.

In the following, we always equip X with the L_∞ norm so that it becomes a Banach space.

Proposition 2.2 *The functional Q_p, $1 \le p \le \infty$ has the following properties:*

i) *Q_p is convex on X.*

ii) *Q_p^p is strictly convex on* $\operatorname{dom} Q_p$ *for $1 < p < \infty$.*

iii) *Q_p is continuous on X.*

Proof: i) For $f, g \in X$ and $\lambda \in [0,1]$ we have to show that

$$Q_p(\lambda f + (1-\lambda)g) \le \lambda Q_p(f) + (1-\lambda) Q_p(g).$$

If one of the values $\mu\{x : f(x) \le 0\}$ or $\mu\{x : g(x) \le 0\}$ is positive, then the assertion is clear. Assume that both values are zero. Then $\mu\{x : \lambda f(x) + (1-\lambda)g(x) \le 0\} = 0$ and it remains to show that

$$\|Q\left(\cdot, \lambda f + (1-\lambda)g\right)\|_p \le \lambda \|Q(\cdot, f)\|_p + (1-\lambda)\|Q(\cdot, g)\|_p.$$

By Proposition 2.1, we obtain

$$0 \le Q\,(x, \lambda f + (1-\lambda) g) \le \lambda Q\,(x,f) + (1-\lambda) Q\,(x,g) \quad \forall x \in \Omega,$$

and hence

$$\|Q\,(\cdot, \lambda f + (1-\lambda)g)\,\|_p \le \|\lambda Q(\cdot,f) + (1-\lambda) Q(\cdot,g)\|_p \le \lambda \|Q(\cdot,f)\|_p + (1-\lambda)\|Q(\cdot,g)\|_p. \quad (3)$$

ii) Let $f, g \in \operatorname{dom} Q_p$ with $f \ne g$ and $\lambda \in (0,1)$. Then $\lambda f + (1-\lambda) g \in \operatorname{dom} Q_p$ and since $\|\cdot\|_p^p$, $1 < p < \infty$, is strictly convex, we obtain together with (3) that

$$\|Q\,(\cdot, \lambda f + (1-\lambda)g)\,\|_p^p \le \|\lambda Q(\cdot,f) + (1-\lambda) Q(\cdot,g)\|_p^p < \lambda \|Q(\cdot,f)\|_p^p + (1-\lambda)\|Q(\cdot,g)\|_p^p.$$

iii) Since Q_p is proper, convex and there exists a non-empty open set of $\operatorname{dom} Q_p$ where Q_p is bounded above by a finite constant, it is continuous over the interior of $\operatorname{dom} Q_p$, see [6, p. 12]. It remains to show for any function f not in the interior of $\operatorname{dom} Q_p$ and any sequence $\{f_n\}_{n \in \mathbb{N}}$ with $\lim\limits_{n \to \infty} \|f - f_n\|_\infty = 0$ that

$$\liminf_{n \to \infty} Q_p(f_n) \ge Q_p(f) \quad \text{and} \quad \limsup_{n \to \infty} Q_p(f_n) \le Q_p(f). \quad (4)$$

For $p = \infty$ a function f not in the interior of $\operatorname{dom} Q_\infty$ has to fulfill $f(x_0) \le 0$ for some $x_0 \in \Omega$. Then the right inequality in (4) follows immediately and the left one by

$$\begin{aligned} \liminf_{n \to \infty} Q_\infty(f_n) &= \liminf_{n \to \infty} \max\left\{\|f_n\|_\infty - 1, \|1/f_n\|_\infty - 1\right\} \\ &\ge \liminf_{n \to \infty} \max\left\{f_n(x_0) - 1, 1/f_n(x_0) - 1\right\} \;=\; \infty \;=\; Q_\infty(f). \end{aligned}$$

Let $1 \le p < \infty$. Assume that $\mu(\Omega_0) > 0$, where $\Omega_0 := \{x : f(x) \le 0\}$. Then it remains to verify the left inequality in (4). If there exists $x_0 \in \Omega_0$ such that $f(x_0) < 0$, then $f(x) \le -\varepsilon < 0$ in a neighborhood $N(x_0)$ of x_0 and there exists $n(\varepsilon)$ such that $f_n(x) < -\varepsilon/2$ for $x \in N(x_0)$ and $n \ge n(\varepsilon)$. But then $\liminf\limits_{n \to \infty} Q_p(f_n) = \infty$ by definition of Q_p. Hence, we can restrict our attention to $f \ge 0$. Since f_n converges uniformly to f, for any $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $|f_n| \le \varepsilon$ on Ω_0. But then $Q_p(f_n) \ge \mu(\Omega_0)(1/\varepsilon - 1)$ for $n \ge n(\varepsilon)$ which goes to infinity as $\varepsilon \to 0$.
Therefore, it remains to consider the case $\mu(\Omega_0) = 0$ and $\mu\{x : f_n(x) \le 0\} = 0$. Then we get by Fatou's lemma [16, p. 17] and since $\lim_{n \to \infty} Q(\cdot, f_n) = Q(\cdot, f)$ a.e. that

$$Q_p^p(f) = \int_\Omega Q(\cdot,f)^p \, d\mu = \int_\Omega \liminf_{n \to \infty} Q(\cdot, f_n)^p \, d\mu \le \liminf_{n \to \infty} \int_\Omega Q(\cdot, f_n)^p \, d\mu = \liminf_{n \to \infty} Q_p^p(f_n),$$

$$Q_p^p(f) = \int_\Omega Q(\cdot,f)^p \, d\mu = \int_\Omega \limsup_{n \to \infty} Q(\cdot, f_n)^p \, d\mu \ge \limsup_{n \to \infty} \int_\Omega Q(\cdot, f_n)^p \, d\mu = \limsup_{n \to \infty} Q_p^p(f_n).$$

This completes the proof. □

For given $f \in X_{>0}$ and $\varphi_j \in X$, $j = 1, \ldots, n$, we want to find a function $\hat f \in \operatorname{span}\{\varphi_j : j = 1, \ldots, n\}$ such that $Q_p(\hat f / f)$ becomes minimal. In other words, we are interested in

$$\inf_{\alpha \in \mathbb{R}^n} Q_p(A\alpha), \quad (5)$$

where $A : \mathbb{R}^n \to X$ denotes the linear mapping

$$A\alpha := \sum_{j=1}^{n} \alpha_j \underbrace{\frac{\varphi_j}{f}}_{\psi_j}$$

onto its range $\mathcal{R}(A) = \operatorname{span}\{\psi_j : j = 1, \ldots, n\}$.

Remark 2.3 *It seems also natural to consider*

$$\hat{\alpha} := \operatorname*{argmin}_{\alpha \in \mathbb{R}^n} \| \ln(f) - \hat{f} \|_p. \tag{6}$$

For $p = \infty$, this problem is equivalent to

$$\hat{\alpha} := \operatorname*{argmin}_{\alpha \in \mathbb{R}^n} Q_\infty(e^{\hat{f}}/f)$$

The approximation with respect to the L_p norm as considered in (6) *is well examined, see* [14] *and the references therein. For a numerical comparison of* (6) *for $p = \infty$ with our approach see Example 3.1.*

Since $\mathcal{R}(A)$ is a finite dimensional linear subspace of X it is closed. By $\mathcal{N}(A)$ we denote the nullspace of A. By the following proposition, $Q_p(A\cdot)$ attains its minimum under mild assumptions on A.

Proposition 2.4 *Let $\mathcal{R}(A) \cap \operatorname{dom} Q_p \neq \emptyset$. Then $Q_p(A\cdot)$, $1 \leq p \leq \infty$ attains its minimum. If $\mathcal{N}(A) = \{0\}$, then, for $1 < p < \infty$, the functional $Q_p(A\cdot)$ has a unique minimizer.*

Proof: The restriction $Q_p|_{\mathcal{R}(A)}$ of Q_p onto the reflexive Banach space $(\mathcal{R}(A), \| \cdot \|_\infty)$ is a proper, convex, lower semi-continuous functional which is in addition coercive since $\|f - 1\|_p \leq Q_p(f)$. Thus, Q_p attains its minimum on $\mathcal{R}(A)$. By definition of $\mathcal{R}(A)$ a corresponding minimizer has the form $A\hat{\alpha}$ for some $\hat{\alpha} \in \mathbb{R}^n$ and this is also a minimizer of $Q_p(A\cdot)$.
For $1 < p < \infty$, the minimizers of Q_p and Q_p^p coincide. Since Q_p^p is strictly convex on $\operatorname{dom} Q_p$, it has a unique minimizer $\hat{v} \in \mathcal{R}(A)$ and since $\mathcal{N}(A) = \{0\}$ this implies that there exists a unique $\hat{\alpha} \in \mathbb{R}^n$ such that $\hat{v} = A\hat{\alpha}$. This completes the proof. □

3 Minimization by second order cone programming

In this section, we deal with the discrete setting, i.e., we consider $\Omega := \{x_1, \ldots, x_m\}$ and $X := \mathbb{R}^m$. Then for $f := (f(x_i))_{i=1}^m \in \mathbb{R}^m_{>0}$ the linear mapping A can be represented by the matrix $A := (\varphi_j(x_i)/f(x_i))_{i,j=1}^{m,n}$. We suppose that $n \leq m$ and that A has full range n so that $\mathcal{N}(A) = \{0\}$. Then, for $p = 1, 2$ and ∞, the problems

$$\hat{\alpha} = \operatorname*{argmin}_{\alpha \in \mathbb{R}^n} Q_p^p(A\alpha), \quad \text{resp.,} \quad \hat{\alpha} = \operatorname*{argmin}_{\alpha \in \mathbb{R}^n} Q_\infty(A\alpha) \tag{7}$$

can be simply solved by *second order cone programming* (SOCP). In general, SOCP can be applied for solving problems of the form

$$\min_{x\in\mathbb{R}^s}\langle c,x\rangle \quad \text{subject to} \quad Mx+b\in K \tag{8}$$

where $c\in\mathbb{R}^s$, $b\in\mathbb{R}^t$, $M\in\mathbb{R}^{t,s}$ and K is the product of convex cones of the form $\mathbb{R}^{\tau}_{\geq 0}$, $\{0\}$ or

$$\mathbf{L}^{\tau} := \{(\bar{x}^{\mathrm{T}},x_\tau)^{\mathrm{T}} = (x_1,\dots,x_\tau)^{\mathrm{T}} : \|\bar{x}\|_2 \leq x_\tau\}$$

$$\mathbf{L}^{\tau}_r := \Big\{\big(\bar{x}^{\mathrm{T}},x_{\tau-1},x_\tau\big)^{\mathrm{T}} = (x_1,\dots,x_\tau)^{\mathrm{T}} : \|\bar{x}\|_2^2 \leq 2\,x_{\tau-1}x_\tau,\ x_{\tau-1}\geq 0\Big\}.$$

Software packages like MOSEK [1] provide efficient large scale solvers for problems of this kind. For details on SOCP we refer to [9]. It remains to rewrite (7) into the form (8). For $p=\infty$, problem (7) is equivalent to the constraint problem

$$\min_{u\in\mathbb{R}^m,\alpha\in\mathbb{R}^n} Q_\infty(u) \quad \text{subject to} \quad A\alpha = u$$

which can be rewritten as

$$\min_{a\in\mathbb{R},u\in\mathbb{R}^m,\alpha\in\mathbb{R}^n} a-1 \qquad \text{subject to} \quad A\alpha=u,\ 1\leq a,\ \frac{1}{a} \leq u\leq a, \tag{9}$$

where the inequalities are meant componentwise. The first two constraints and $u\leq a$ are cone constraints with $K=\{0\}$ or $\mathbb{R}^t_{\geq 0}$. The remaining constraints $1\leq au_i$ are equivalent to $\sqrt{2^2+(a-u_i)^2}\leq u_i+a$, $i=1,\dots,m$ and can therefore be reformulated as

$$\begin{pmatrix} 0 & 0\\ -1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix} u_i\\ a\end{pmatrix} + \begin{pmatrix} 2\\ 0\\ 0\end{pmatrix} \in \mathbf{L}^3.$$

For $p=1$, problem (7) can be rewritten as

$$\min_{u\in\mathbb{R}^m,\alpha\in\mathbb{R}^n}\sum_{i=1}^m \left|\max\{u(x_i),\frac{1}{u(x_i)}\}-1\right| \quad \text{subject to} \quad A\alpha=u,\ 0<u$$

and in SOCP–form as

$$\min_{a,u\in\mathbb{R}^m,\alpha\in\mathbb{R}^n}\sum_{i=1}^m a_i - m \qquad \text{subject to} \quad A\alpha=u,\ 1\leq a,\ \frac{1}{a}\leq u\leq a.$$

For $p=2$, problem (7) is equivalent to

$$\min_{u\in\mathbb{R}^m,\alpha\in\mathbb{R}^n}\sum_{i=1}^m \left(\max\{u(x_i),\frac{1}{u(x_i)}\}-1\right)^2 \quad \text{subject to} \quad A\alpha=u,\ 0<u$$

and further to

$$\min_{a,b,c,u\in\mathbb{R}^m,\,\alpha\in\mathbb{R}^n}\sum_{i=1}^m c_i \quad \text{subject to} \quad A\alpha=u,\ 1\leq a,\ \frac{1}{a}\leq u\leq a,\ b=a-1,\ b^2\leq c.$$

As in the previous problem these are second order cone constraints, where the fifth condition is related to a rotated second order cone with $(b_i, c_i, 1/2)^{\mathrm{T}} \in \mathbf{L}_r^3$, $i = 1, \ldots, m$.

We finish this section by an example. Since our original motivation to deal with this topic comes from query optimization in relational database management systems we give an example with data from this area.

Example 3.1 *The dots in Fig. 2 show the number of authors for a given number of citations between 256 and 512 times as extracted from the citeseer top 10.000 cited computer science authors. The solid lines are the approximations of the data by polynomials of degree 1,2 and 4 with respect to* Q_p *for* $p = 1, 2, \infty$. *The dashed lines in the bottom figures show the approximations by* $e^{\hat{f}}$ *from problem* (6) *in Remark 2.3, with* $p = \infty$, *where* $\hat{f}$ *is again a polynomial of degree 1,2 and 4. The corresponding minimal values of* $Q_p(\hat{f}/f)$ *are given in the following table. The last column of the table shows the value* $\max_i \max\{f(x_i)/e^{\hat{f}(x_i)} - 1, e^{\hat{f}(x_i)}/f(x_i) - 1\}$ *for the approximation* (6) *with* $p = \infty$.

degree	Q_1	Q_2	Q_∞	Q_∞, *exp*
1	64.3062	30.2001	1.1606	*1.1077*
2	60.6107	25.5951	0.9740	*1.0448*
3	60.5563	25.5942	0.9700	*0.9957*
4	60.4704	25.5163	0.9321	*0.9493*

4 The Q_∞ functional

In this section, we have a closer look at the Q_∞ functional. In particular, we are interested in conditions on $A : \mathbb{R}^n \to X$ such that the minimizer of $Q_\infty(A\cdot)$ is unique. Let X' denote the dual space of X. Of course $(R^m)' = R^m$, while the dual space of $(C(\Omega), \|\cdot\|_\infty)$ is the Banach space $M(\Omega)$ of regular (signed) Borel measures equipped with the total variation. Note that we know by the Krein–Milman theorem and the theorem of Alaoglu [15, Sec. VIII] that

$$\{p \in M(\Omega) : \|p\| \le 1\} = \overline{\mathrm{conv}}\,\{\xi_x \delta(x) : |\xi_x| = 1,\ x \in \Omega\}\,,$$

where $\langle \delta(x), f\rangle = v(x)$ for all $f \in C(\Omega)$ and $\overline{\mathrm{conv}}$ denotes the closure of the convex hull in the weak* topology of X'.
In the following, we assume that $\mathcal{R}(A) \cap \mathrm{dom}\, Q_\infty \neq \emptyset$ such that a minimizer of $Q_\infty(A\cdot)$ exists. Note that $\mathrm{dom}\, Q_\infty = X_{>0}$. Further, we see that there exists $u \in \mathcal{R}(A)$ with

$$Q_\infty(u) = 0 \quad \Leftrightarrow \quad u \equiv 1 \quad \Leftrightarrow \quad 1 \in \mathcal{R}(A),$$

so that we restrict our attention to the nontrivial case $Q_\infty(u) > 0$.
The subdifferential $\partial Q_\infty(u)$ of the proper convex functional Q_∞ at $u \in X_{>0}$ is defined as

$$\partial Q_\infty(u) := \{p \in X' : Q_\infty(u) \le Q_\infty(v) + \langle p, u - v\rangle\ \forall v \in X\}. \tag{10}$$

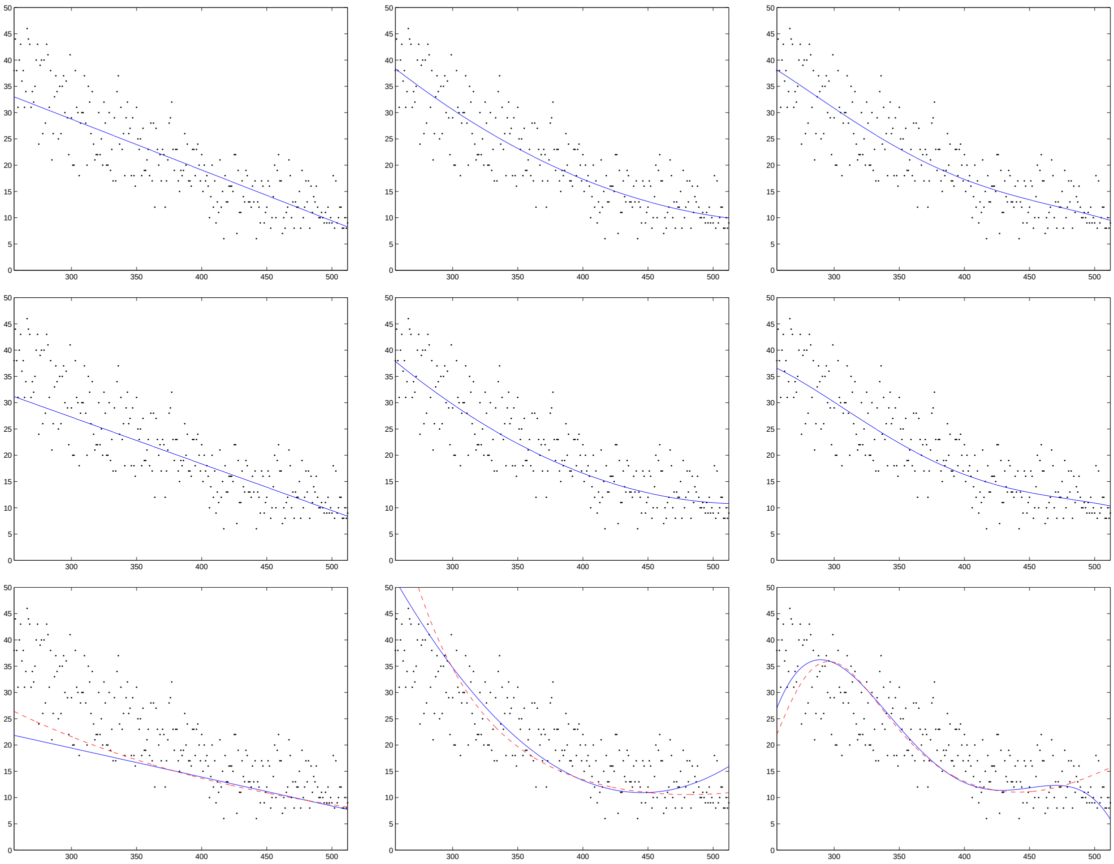

Figure 2: Approximation by polynomials of degree 1, 2 and 4 (left to right). Top: with respect to Q_1. Middle: with respect to Q_2. Bottom: with respect to Q_∞ and $e^{\hat{f}}$ for $\hat{f}$ approximated by (6) with $p = \infty$ (dashed line).

By Fermat's rule we know that $\hat{\alpha}$ is a minimizer of $\tilde{Q} := Q_\infty(A\cdot)$ if and only if

$$0 \in \partial \tilde{Q}(\hat{\alpha}) = A^* \partial Q_\infty(A\hat{\alpha}). \tag{11}$$

Therefore we are interested in ∂Q_∞. We will show that $\partial Q_\infty(u)$ is the weak* closure of certain linear combinations of Dirac measures. To this end, we need the following theorem. The proof can be found, e.g., in [10, pp. 201].

Theorem 4.1 *Let Ω be a compact topological space and let X be a separable locally convex topological space. Let $F(x, u)$ be a function on $\Omega \times X$ which is upper semi–continuous in x for every $u \in X$ and convex in u for every $x \in \Omega$. Set $G(u) := \sup_{x\in\Omega} F(x, u)$. If $F(x, \cdot)$ is continuous at u for any $x \in \Omega$, then*

$$\partial G(u) = \overline{\mathrm{conv}} \left\{ \partial_u F(x, u) : x \in \Omega,\ F(x, u) = G(u) \right\}.$$

This theorem can be used to prove the following theorem.

Theorem 4.2 *Let $u \in X_{>0}$ with $Q_\infty(u) > 0$ and let*

$$E = E(u) := \{x \in \Omega : Q(x,u) = Q_\infty(u)\} \tag{12}$$

be the extremal set *of u. Then the subdifferential of Q_∞ at u is given by*

$$\partial Q_\infty(u) = \overline{\mathrm{conv}} \left\{ (Q_\infty(u)+1)^{1-\theta_x}\, \theta_x\, \delta(x) : x \in E \right\},$$

where $\theta_x := \mathrm{sgn}(u(x) - 1)$.

Proof: Let $a := Q_\infty(u) + 1 > 1$. By Proposition 2.1 and Theorem 4.1 with $F(x,u) := Q(x,u)$ and $G := Q_\infty$ it remains to show that

$$\partial_u Q(x,u) = a^{1-\theta_x}\theta_x\delta(x) = \begin{cases} \delta(x) & \text{if } u(x) = a, \\ -a^2\delta(x) & \text{if } u(x) = 1/a, \end{cases} \qquad x \in E.$$

Let $x \in E$ and $p \in \partial Q_u(x,u)$. Then p has to fulfill

$$Q(x,u) \le Q(x,v) + \langle p, u - v\rangle \quad \forall v \in X. \tag{13}$$

Set $v := u \pm h$, $h \in X$, where $h(x) = 0$ so that $Q(x,u) = Q(x,v) = a - 1$. Then (13) implies for any $h \in X$ with $h(x) = 0$ that

$$0 \le \pm\langle p, h\rangle \quad \Leftrightarrow \quad \langle p, h\rangle = 0.$$

Consequently, p is supported on x, i.e., $p = c\delta(x)$.
If $a = u(x)$, then (13) implies

$$a - 1 \le Q(x,v) + c(a - v(x)) \quad \forall v \in X$$

and choosing $v \in X$ such that $v(x) \ge 1$ we obtain

$$\begin{aligned} a - 1 &\le v(x) - 1 + c(a - v(x)), \\ 0 &\le (1-c)\,(v(x) - a)\,. \end{aligned}$$

Choosing $v \in X$ such that $v(x) > a$ and then such that $v(x) < a$, this implies that $c = 1$.
If $1/u(x) = a$, then (13) can be rewritten as

$$a - 1 \le Q(x,v) + c\left(\frac{1}{a} - v(x)\right) \quad \forall v \in X$$

and for $v \in X$ with $v(x) < 1$ we get

$$\begin{aligned} a - 1 &\le \frac{1}{v(x)} - 1 + c\left(\frac{1}{a} - v(x)\right), \\ 0 &\le (a + cv(x))\,(1 - av(x))\,. \end{aligned}$$

In the case $v(x) < 1/a$, this implies that $a + cv \ge 0$, i.e., $c \ge -a^2$. Choosing $v \in X$ such that $v(x) > 1/a$ we conclude that $c \le -a^2$ so that finally $c = -a^2$. This completes the proof. □

The following theorem characterizes the minimizers $\hat{\alpha} = \mathrm{argmin}\, Q_\infty(A\cdot)$.

Theorem 4.3 *Let* $A : \mathbb{R}^n \to X$ *be given by* $A\alpha := \sum_{j=1}^n \alpha_j \psi_j$, $\psi_j \in X$, *where* $\mathcal{R}(A) \cap X_{>0} \neq \emptyset$ *and* $1 \notin \mathcal{R}(A)$. *Assume that* $\mathcal{R}(A)$ *contains only functions* u *for which the set* $E(u)$ *defined by* (12) *is finite. Then*

$$\hat{\alpha} = \operatorname*{argmin}_{\alpha \in \mathbb{R}^n} Q_\infty(A\alpha) \quad \text{and} \quad \hat{u} = A\hat{\alpha}, \quad \hat{a} = Q_\infty(A\hat{\alpha}) + 1 \tag{14}$$

if and only if there exist $\hat{\lambda} \in \mathbb{R}^t$, $t \leq n+1$ *with* $\hat{\lambda}_i \neq 0$, $i = 1, \ldots, t$ *and* $\hat{x}_i \in E(\hat{u})$, $i = 1, \ldots, t$ *such that*

i) $\sum\limits_{i=1}^t \hat{\lambda}_i \, \psi_j(\hat{x}_i) = 0, \quad j = 1, \ldots, n,$

ii) $\sum\limits_{i=1}^t \hat{\lambda}_i \, \theta_{\hat{x}_i} \hat{a}^{\theta_{\hat{x}_i} - 1} = 1, \quad \theta_{\hat{x}_i} := \operatorname{sgn}(\hat{u}(\hat{x}_i) - 1),$

iii) *if* $\hat{\lambda}_i > 0$ *then* $\hat{u}(\hat{x}_i) = \hat{a}$ *and if* $\hat{\lambda}_i < 0$ *then* $\hat{u}(\hat{x}_i) = \frac{1}{\hat{a}}$.

Proof: By (11), we have that $\hat{\alpha}$ is a minimizer of $Q_\infty(A\cdot)$ if and only if there exists $\hat{p} \in \partial Q_\infty(A\hat{\alpha})$ such that $0 = A^*\hat{p} = (\langle \hat{p}, \psi_j \rangle)_{j=1}^n$. By Theorem 4.2 we know that $\hat{p}$ has the form

$$\hat{p} = \sum_{x_i \in E(\hat{u})} \mu_i \, \hat{a}^{1-\theta_{x_i}} \theta_{x_i} \delta(x_i)$$

with $\mu_i \geq 0$, $\sum_i \mu_i = 1$. Thus, $\hat{\alpha}$ is a minimizer of $Q_\infty(A\cdot)$ if and only if

$$0 = \sum_{x_i \in E(\hat{u})} \mu_i \, \hat{a}^{1-\theta_{x_i}} \theta_{x_i} \psi_j(x_i), \qquad j = 1, \ldots, n.$$

In other words, 0 is a convex combination of the n-dimensional vectors $\left(\hat{a}^{1-\theta_{x_i}} \theta_{x_i} \psi_j(x_i)\right)_{j=1}^n$. By Carathéodory's theorem we know that for any subset $D \subset \mathbb{R}^n$, any point of $\operatorname{conv}(D)$ can be expressed as a convex linear combination of $t \leq n+1$ points of D. Consequently, there exist $t \leq n+1$ points $\hat{x}_i$ from $E(\hat{u})$ and $\hat{\mu}_i > 0$, $\sum_{i=1}^t \hat{\mu}_i = 1$ such that

$$0 = \sum_{i=1}^t \underbrace{\hat{\mu}_i \, \hat{a}^{1-\theta_{\hat{x}_i}} \theta_{\hat{x}_i}}_{\hat{\lambda}_i} \psi_j(\hat{x}_i), \qquad j = 1, \ldots, n.$$

We have that $\hat{\lambda}_i \, \theta_{\hat{x}_i} > 0$ and $\sum_{i=1}^t \hat{\mu}_i = \sum_{i=1}^t \hat{\lambda}_i \, \hat{a}^{\theta_{\hat{x}_i} - 1} \theta_{\hat{x}_i} = 1$. If $\hat{\lambda}_i > 0$ then $\theta_{\hat{x}_i} = 1$ and $\hat{u}(\hat{x}_i) = \hat{a}$ by definition of $E(\hat{u})$. Conversely, if $\hat{\lambda}_i < 0$ then $\theta_{\hat{x}_i} = -1$ and $\hat{u}(\hat{x}_i) = 1/\hat{a}$. This finishes the proof. □

Corollary 4.4 *Let the assumptions of Theorem* 4.3 *be fulfilled. Let*

$$\hat{\alpha} = \operatorname*{argmin}_{\alpha \in \mathbb{R}^n} Q_\infty(A\alpha), \quad \hat{u} = A\hat{\alpha}, \quad \hat{a} = Q_\infty(A\hat{\alpha}) + 1$$

and let $\hat{x}_i \in E(\hat{u})$, $i = 1, \ldots, t$ *denote the points in* i) - iii) *of Theorem* 4.3. *Then, any other minimizer* $\tilde{\alpha}$ *of* $Q_\infty(A\cdot)$ *and* $\tilde{u} = A\tilde{\alpha}$ *fulfill*

$$\tilde{u}(\hat{x}_i) = \hat{u}(\hat{x}_i) = \hat{a}^{\theta_{\hat{x}_i}}.$$

Proof: By Theorem 4.3 there exist $\hat\lambda \in \mathbb{R}^t$ such that

$$\sum_{i=1}^t \hat\lambda_i \, \psi_j(\hat x_i) = 0, \quad j = 1, \ldots, n$$

and $\hat\lambda_i \theta_{\hat x_i} > 0$. Taking this into account we obtain

$$\begin{aligned} \sum_{i=1}^t |\hat\lambda_i| \, |\hat a^{\theta_{\hat x_i}} - 1| &= \sum_{i=1}^t \hat\lambda_i \, (\hat u(\hat x_i) - 1) \\ &= \sum_{i=1}^t \hat\lambda_i \sum_{j=1}^n \hat\alpha_j \psi_j(\hat x_i) - \sum_{i=1}^t \hat\lambda_i \\ &= \sum_{j=1}^n \hat\alpha_j \sum_{i=1}^t \hat\lambda_i \psi_j(\hat x_i) - \sum_{i=1}^t \hat\lambda_i = -\sum_{i=1}^t \hat\lambda_i \\ &= \sum_{i=1}^t \hat\lambda_i \, (\tilde u(\hat x_i) - 1) = \sum_{i=1}^t |\hat\lambda_i| \, \theta_{\hat x_i} \, (\tilde u(\hat x_i) - 1). \end{aligned}$$

For those $\hat x_i$, $i = 1, \ldots, t$ with $\theta_{\hat x_i} = \mathrm{sgn}(\tilde u(\hat x_i) - 1)$ we have that $|\tilde u(\hat x_i) - 1| \le |\hat a^{\theta_{\hat x_i}} - 1|$. Then we get for the remaining indices in $I := \{i = 1, \ldots, t : \theta_{\hat x_i} \ne \mathrm{sgn}(\tilde u(\hat x_i) - 1)\}$ that

$$\sum_{i \in I} |\hat\lambda_i| \, |\hat a^{\theta_{\hat x_i}} - 1| \le \sum_{i \in I} |\hat\lambda_i| \, \theta_{\hat x_i} \, (\tilde u(\hat x_i) - 1) \le 0.$$

Since the left-hand side is positive, this implies that I is empty and that $\hat u(\hat x_i) = \hat a^{\theta_{\hat x_i}}$. □

Now we can address the question of the uniqueness of the minimizer.
First, we consider the discrete setting $X = \mathbb{R}^m$ with

$$A := (\varphi_j(x_i)/f(x_i))_{i,j=1}^{m,n} = (\psi_j(x_i))_{i,j=1}^{m,n} \,. \tag{15}$$

By spark(A) we denote the smallest number of rows of A which are linearly dependent. In other words, any spark$(A) - 1$ rows of A are linearly independent. For the 'spark' notation we also refer to [5].

Theorem 4.5 *Let $A \in \mathbb{R}^{m,n}$, $m \ge n$ such that* spark$(A) = n + 1$. *Then $Q_\infty(A\cdot)$ has a unique minimizer which is determined by $n + 1$ rows of A, i.e., there exists a set $\hat E \subset \{x_1, \ldots, x_m\}$ of cardinality $|\hat E| = n + 1$ such that $Q_\infty(A\cdot)$ and $Q_\infty(A|_{\hat E}\cdot)$ have the same minimum and the same minimizer. Here $A|_{\hat E}$ denotes the restriction of A to the rows belonging to $\hat E$.*

Proof: Let $\hat E := \{\hat x_i : i = 1, \ldots, t\}$, $t \le n + 1$ denote the points in Theorem 4.3. Then we have by i) of Theorem 4.3 that $(A|_{\hat E})^* \hat\lambda = 0$. If $t \le n$, this implies by spark$(A) = n + 1$ the contradiction $\hat\lambda = 0$. Thus, $t = n + 1$. In particular, if $m = n + 1$, then x_i and $\hat x_i$, $i = 1, \ldots, n + 1$ coincide.

Assume now that there exist two different minimizers $\hat{\alpha}$ and $\tilde{\alpha}$ of $Q_\infty(A\cdot)$. Then we conclude by Corollary 4.4 that $A|_{\hat{E}}(\hat{\alpha}-\tilde{\alpha}) = 0$. Since $A|_{\hat{E}} \in \mathbb{R}^{n+1,n}$ has full rank this is only possible if $\hat{\alpha} = \tilde{\alpha}$.
Similarly, if $\hat{\beta}$ is a minimizer of $Q_\infty(A|_{\hat{E}}\cdot)$, then Corollary 4.4 implies that $A|_{\hat{E}}(\hat{\alpha}-\hat{\beta}) = 0$, i.e., $\hat{\alpha} = \hat{\beta}$ and we are done. □

Remark 4.6 *In general the condition* spark$(A) = n+1$ *is not necessary for* $Q_\infty(A\cdot)$ *to have a unique minimizer. However, if* $A \in \mathbb{R}^{n+1,n}$ *and* $\mathcal{R}(A) \cap \mathbb{R}^{n+1}_{>0} \neq \emptyset$, *then* spark$(A) = n+1$ *is also necessary for* $Q_\infty(A\cdot)$ *to have a unique minimizer.*

Next, we consider the continuous setting with

$$A\alpha := \sum_{j=1}^{n} \alpha_j \varphi_j(x)/f(x), \quad f > 0. \tag{16}$$

A set of continuous functions $\varphi_j : \Omega \to \mathbb{R}$, $j = 1,\ldots,n$ is called a *Chebyshev set* or a *Haar set*, if every non-trivial linear combination of these functions has at most $n-1$ zeros in Ω. In other words, for any collection of n pairwise distinct points $x_i \in \Omega$, the matrix $(\varphi_j(x_i))_{i,j=1}^n$ and the matrix $\mathrm{diag}(1/f(x_i))_{i=1}^n \, (\varphi_j(x_i))_{i,j=1}^n = (\varphi_j(x_i)/f(x_i))_{i,j=1}^n$ is invertible. In particular, in this case the matrix (15) fulfills spark$(A) = n+1$. Of course, depending on the points x_i, the condition spark$(A) = n+1$ can be also fulfilled if $\{\varphi_j : j = 1,\ldots,n\}$ is not a Chebyshev set. For an interval $\Omega = I \subset \mathbb{R}$, the set of polynomials $\varphi_i(x) = x^{i-1}$, $i = 1,\ldots,n$ forms a Chebyshev set. Unfortunately, for $\Omega \subset \mathbb{R}^d$, $d \geq 2$ there does not exist a Chebyshev set of $n \geq 1$ continuous functions.

Theorem 4.7 *Let the functions* $\varphi_j : I \to \mathbb{R}$, $j = 1,\ldots,n$ *form a Chebyshev set and let* A *be defined by* (16). *Then the minimizer of* $Q_\infty(A\cdot)$ *is unique and is determined by the solution of the corresponding discrete problem at* $n+1$ *points of* I.

Proof: Let $\hat{E} := \{\hat{x}_i : i = 1,\ldots,t\}$ denote the points in Theorem 4.3. Then we have by i) of Theorem 4.3 that $(A|_{\hat{E}})^*\hat{\lambda} = 0$. Since $\{\varphi_j\}_{j=1}^n$ is a Chebyshev set, this implies for $t \leq n$ the contradiction $\hat{\lambda} = 0$. Thus, $t = n+1$ and the rest of the proof follows as in the proof of Theorem 4.5. □

Similarly as the best approximating function from the span of a Chebyshev set with respect to $\|\cdot\|_\infty$, the minimizing function $\hat{u} = A\hat{\alpha}$ of $Q_\infty(A\cdot)$ shows an alternating behavior in the $n+1$ points $\hat{x}_i$.

Theorem 4.8 *Let* $\varphi_j : I \to \mathbb{R}$, $j = 1,\ldots,n$ *form a Chebyshev set and let* $\hat{x}_1 < \ldots < \hat{x}_{n+1}$ *denote a set of points fulfilling* i) - iii) *of Theorem* 4.3. *Let* A *be defined by* (15) *or* (16). *Then the components of the corresponding vector* $\hat{\lambda} \in \mathbb{R}^{n+1}$ *have alternating signs. In other words, the values* $\hat{u}(\hat{x}_i) = \hat{f}(\hat{x}_i)/f(\hat{x}_i)$ *coincide alternatingly with* $\hat{a} := \min_\alpha Q_\infty(A\alpha) + 1$ *and* $1/\hat{a}$.

Conversely, if there exists $c > 0$ and $\tilde{\alpha} \in \mathbb{R}^n$ such that

$$\sum_{j=1}^{n} \tilde{\alpha}_j \varphi_j(\hat{x}_i)/f(\hat{x}_i) = c^{(-1)^i}, \quad i = 1, \ldots, n+1, \tag{17}$$

then $\max\{c, 1/c\} = \hat{a}$ *and* $\tilde{\alpha} = \operatorname{argmin}_\alpha Q_\infty(A\alpha)$.

Using Theorem 4.3 the proof follows similarly as for the $\| \cdot \|_\infty$ approximation, see [14]. We add the proof for convenience.

Proof: Let $\Phi \in \mathbb{R}^{n+1,n}$ and $A \in \mathbb{R}^{n+1,n}$ be defined by

$$\Phi := (\varphi_j(\hat{x}_i))_{i,j=1}^{n+1,n} = \begin{pmatrix} \phi_1^{\mathrm{T}} \\ \vdots \\ \phi_{n+1}^{\mathrm{T}} \end{pmatrix} \quad \text{and} \quad A := \underbrace{\operatorname{diag}(1/f(\hat{x}_i))_{i=1}^{n+1}}_{D} \Phi = \begin{pmatrix} a_1^{\mathrm{T}} \\ \vdots \\ a_{n+1}^{\mathrm{T}} \end{pmatrix}.$$

By $\Phi_i, A_i \in \mathbb{R}^{n,n}$ we denote the matrices obtained from Φ, A by cancelling their i-th row. By Theorem 4.3 i) we know that

$$0 = A^{\mathrm{T}}\hat{\lambda} = \Phi^{\mathrm{T}} \underbrace{D\hat{\lambda}}_{\hat{\mu}} \quad \Leftrightarrow \quad \Phi_{n+1}^{\mathrm{T}}(\hat{\mu}_1, \ldots, \hat{\mu}_n)^{\mathrm{T}} = -\hat{\mu}_{n+1}\phi_{n+1}. \tag{18}$$

Since $f > 0$ the components of $\hat{\mu}$ have the same signs as those of $\hat{\lambda}$. Then it follows by Cramer's rule that

$$\hat{\mu}_i = \frac{1}{\det \Phi_{n+1}} \det(\phi_1, \ldots, \phi_{i-1}, -\hat{\mu}_{n+1}\phi_{n+1}, \phi_{i+1}, \ldots, \phi_n) = -\hat{\mu}_{n+1} \frac{(-1)^{n-i} \det \Phi_i}{\det \Phi_{n+1}}.$$

Because $\{\varphi_i\}_{i=1}^n$ is a Chebyshev set, $\operatorname{sgn}(\det \Phi_i)$ coincides for all $i = 1, \ldots, n+1$, see [14, p. 55] and we obtain the first assertion.
Conversely, assume that (17) is fulfilled. Then we have that
$c^{(-1)^{n+1}} = a_{n+1}^{\mathrm{T}}\tilde{\alpha} = a_{n+1}^{\mathrm{T}} A_{n+1}^{-1} \left(c^{(-1)^i}\right)_{i=1}^n$. On the other hand, we obtain by (18) that

$$\begin{aligned} A_{n+1}^{\mathrm{T}}(\hat{\lambda}_1, \ldots, \hat{\lambda}_n)^{\mathrm{T}} &= -\hat{\lambda}_{n+1} a_{n+1}, \\ (\hat{\lambda}_1, \ldots, \hat{\lambda}_n) &= -\hat{\lambda}_{n+1} a_{n+1}^{\mathrm{T}} A_{n+1}^{-1}, \\ -(\hat{\lambda}_1, \ldots, \hat{\lambda}_n)/\hat{\lambda}_{n+1} &= a_{n+1}^{\mathrm{T}} A_{n+1}^{-1} \end{aligned}$$

so that

$$c^{(-1)^{n+1}} = -\frac{1}{\hat{\lambda}_{n+1}}(\hat{\lambda}_1, \ldots, \hat{\lambda}_n)\left(c^{(-1)^i}\right)_{i=1}^n = -\frac{1}{c}\frac{1}{\hat{\lambda}_{n+1}} \sum_{l=1}^{\lceil n/2 \rceil} \hat{\lambda}_{2l-1} - c\frac{1}{\hat{\lambda}_{n+1}} \sum_{l=1}^{\lfloor n/2 \rfloor} \hat{\lambda}_{2l}.$$

However, $c > 0$ is uniquely determined by this equation which is also fulfilled by $\hat{a}$ or $1/\hat{a}$. The rest of the assertion follows by the uniqueness of the minimizer. □

Let $x_1, \dots, x_{n+1} \in I$ be pairwise distinct points. To find the unique polynomial $\hat{f} \in \Pi_{n-1} = \operatorname{span}\{x^{i-1} : i = 1, \dots, n\}$ with the property that

$$\frac{\hat{f}(x_i)}{f(x_i)} = c^{(-1)^i}, \quad i = 1, \dots, n+1$$

for some $c > 0$, Dahmen [4] proposed the following method, compare [14, p. 79] for the ordinary $\|\cdot\|_\infty$ approximation: compute the interpolating polynomials $p, q \in \Pi_n$ corresponding to the knots $(x_i, f(x_i))$, $i = 1, \dots, n+1$ and $(x_i, g(x_i))$, $i = 1, \dots, n+1$, resp., where $g(x_i) := (-1)^{i-1} f(x_i)$. The leading coefficients a_n of p and b_n of q are the divided differences $a_n = f[x_1, \dots, x_{n+1}]$ and $b_n = g[x_1, \dots, x_{n+1}]$. We know that $b_n \neq 0$, since there doesn't exist a polynomial in Π_{n-1} with n zeros. It is not hard to show that $|a_n| \neq |b_n|$. If $a_n = 0$ we are done and $\hat{f} = p$. If $|a_n| < |b_n|$, we set

$$\hat{f} := (p - \varepsilon\, q)/\sqrt{1 - \varepsilon^2}, \quad \text{where} \quad \varepsilon := a_n/b_n \in (-1, 1).$$

By construction we have that $\hat{f} \in \Pi_{n-1}$ and $\hat{f}(x_i) = f(x_i)(1 + (-1)^i \varepsilon)/\sqrt{1 - \varepsilon^2}$, i.e.

$$\frac{\hat{f}(x_i)}{f(x_i)} = \sqrt{\frac{1 + (-1)^i \varepsilon}{1 - (-1)^i \varepsilon}} = c^{(-1)^i}, \qquad c := \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}}.$$

If $|a_n| > |b_n|$, we change to roles of p and q.
This method can be generalized for other Chebyshev sets but is less efficient if we have no analog to fast polynomial interpolation methods.

Based on the computation of the best Q_∞ approximation with respect to $n+1$ points (by the above method or SOCP) we can modify known methods from ordinary best $\|\cdot\|_\infty$ approximation to find the overall best Q_∞ approximation. We have only to be careful with negative function evaluations which may appear in the algorithm. They can be handled by ideas from the following remark. In the discrete case, an ascending (or descending) algorithm can be applied and in the continuous case Remes-type algorithms, see [14] or also the algorithm in [8].

Remark 4.9 *In* [8], *the univariate best approximation with respect to the* generalized relative error

$$\|(f - \hat{f})/\max\{|f|, |\hat{f}|\}\|_\infty \tag{19}$$

for linear combinations $\hat{f}$ *of a Chebyshev set was considered and a linear Remes-type algorithm was proposed. The algorithm is based on an* alternation theorem *which was announced to be in a submitted paper. To the best of our knowledge, this paper has never been published. In contrast to our functional which reads*

$$\|\max\{f/\hat{f}, \hat{f}/f\} - 1\|_\infty = \|(f - \hat{f})/\min\{f, \hat{f}\}\|_\infty, \quad f, \hat{f} > 0 \tag{20}$$

the functional in (19) *is not convex in* $\hat{f}$. *Using quotients with* $y = \hat{f}(x)/f(x)$ *the point evaluations in* (19) *read*

$$\tilde{q}(y) = \begin{cases} 1 - y & \text{for } |y| \leq 1, \\ 1 - \frac{1}{y} & \text{for } |y| > 1 \end{cases} \tag{21}$$

instead of (1)*. Note that* $\tilde{q}(y) \geq 1$ *for* $y \leq 0$*. For* $f > 0$ *both functionals* (19) *and* (20) *have the same minimizer which can be seen as follows: the function* $\hat{f}$ *minimizes our functional*

$$\|\max\{f/\hat{f}, \hat{f}/f\}\|_\infty \quad = \quad \|1/\min\{\hat{f}/f, f/\hat{f}\}\|_\infty, \quad \hat{f} > 0$$

if and and only if it minimizes

$$\|(f - \hat{f})/\max\{f, \hat{f}\}\|_\infty \quad = \quad \|1 - \min\{\hat{f}/f, f/\hat{f}\}\|_\infty$$

as long as the minimizer of (19) *is indeed positive. This is always the case by the following argument: Let* $\hat{f} > 0$ *be the minimizer of our functional* (20) *and* $\hat{a} := Q_\infty(\hat{f}) + 1$*. Then* $1 - \min\{\hat{f}(x)/f(x), f(x)/\hat{f}(x)\} \leq 1 - 1/\hat{a}$ *for all* $x \in I$*. Assume that there exists a minimizer* $\tilde{f}$ *of* (19) *with* $\tilde{f}(\tilde{x}) \leq 0$ *for some* $\tilde{x} \in I$*. But then, by* (21)*, we have* $|f(\tilde{x}) - \hat{f}(\tilde{x})|/\max\{|f(\tilde{x})|, |\hat{f}(\tilde{x})|\} \geq 1$ *such that* $\tilde{f}$ *cannot be a minimizer. Thus, for* $f > 0$*, any minimizer of* (19) *is automatically positive.*
Since for $f > 0$ *both functionals* (19) *and* (20) *have the same minimizer, our* convex approach *proves also the alternation theorem for the best approximation with respect to the generalized relative error. Conversely, for computations one can alternatively use the error measure* (19)*.*

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