# Local Convergence of the Heavy-ball Method and iPiano for Non-convex Optimization

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**Abstract**

A local convergence result for abstract descent methods is proved. The sequence of iterates is attracted by a local (or global) minimum, stays in its neighborhood and converges. This result allows algorithms to exploit local properties of the objective function: The gradient of the Moreau envelope of a prox-regular functions is locally Lipschitz continuous and expressible in terms of the proximal mapping. We apply these results to establish relations between an inertial forward–backward splitting method (iPiano) and inertial averaged/alternating proximal minimization.

***Keywords*** — *nonconvex optimization, inertial forward–backward splitting, non-convex feasibility, prox-regularity, gradient of Moreau envelopes*

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## 1 Introduction

We study the local convergence of an inertial proximal algorithm for non-convex optimization (iPiano) [29, 27]. It is an inertial forward–backward splitting method that generalizes the Heavy-ball method [32, 36]. Related inertial algorithms are analysed in [23, 7, 6]. Based on a convergence result for abstract descent methods, Ochs et al. [29] established iPiano's global convergence to a stationary point. In this paper, the following local convergence result for abstract descent methods is proved: A sequence that starts sufficiently close to a local (or global) minimizer, remains in a neighborhood of this minimizer and converges to the local optimum. The local convergence of iPiano follows as a direct consequence, under conditions that must be satisfied only on this neighborhood.

In particular, this allows us to minimize objective functions that involve the Moreau envelope of prox-regular functions, e.g. the distance function of a prox-regular set. In this particular setting iPiano reveals connections to inertial averaged proximal minimization or alternating proximal minimization, e.g., the inertial averaged/alternating projection method [35] for non-convex feasibility problems [21, 20, 2, 12]. For the problem of finding a point in the intersection of a non-convex and a convex set, global convergence can be established for the inertial methods. These connections rely on the fact that the gradient of the Moreau envelope $e_\lambda f$ of prox-regular functions $f$ with parameter $\lambda > 0$ is locally well defined, Lipschitz continuous, and expressible using the proximal mapping $P_\lambda f$—a result which is well known for convex functions.

If we apply the Heavy-ball method to the minimization of the Moreau envelope of a function, for which the relation $\nabla e_\lambda f(x) = \frac{1}{\lambda}(x - P_\lambda f(x))$ holds along the iterations, an inertial proximal minimization method

$$x^{k+1} = (1-\theta)x^k + \theta P_\lambda f(x^k) + \beta(x^k - x^{k-1}) \qquad (\ \beta \in [0,1)\ )$$

with step size $\theta \in (0,1]$ can be recovered. Applied to the minimization of the sum of Moreau envelopes, it yields an inertial averaged proximal minimization

$$x^{k+1} = (1-\theta)x^k + \frac{\theta}{M}\sum_{i=1}^{M} P_\lambda f_i(x^k) + \beta(x^k - x^{k-1})\,.$$

Applying the inertial forward–backward splitting (iPiano) to the sum of a non-convex function $g$ and the Moreau envelope of a prox-regular function $f$ yields an inertial alternating proximal minimization method

$$x^{k+1} = P_\lambda g\left((1-\theta)x^k + \theta P_\lambda f(x^k) + \beta(x^k - x^{k-1})\right)\,.$$

In case of a non-convex function $g$ and convex function $f$, global convergence is established. Of course, for $\beta = 0$ the inertial term vanishes, and the Heavy–ball method reduces to the gradient descent method and iPiano to forward–backward splitting [24, 14, 8, 11, 2, 10, 13, 22]. These relations, and thus convergence, rely only on the fact that the above-mentioned formula for the gradient of the Moreau envelope can be used and $\nabla e_\lambda f$ is Lipschitz continuous. This may happen to be true globally, if $f$ is convex, or locally, for instance, when $f$ is prox-regular.

Prox-regular functions are a certain class of functions, which was introduced in [31] and comprises primal-lower-nice (introduced by Poliquin [30]), lower-$\mathcal{C}^2$, strongly amenable (see for instance [33]), and proper lower semi-continuous convex functions. It is known that prox-regular functions (locally) share some favorable properties of convex functions. Indeed a function is prox-regular if and only if there exists an ($f$-attentive) localization of the subgradient mapping that is monotone up to a multiple of the identity mapping [31]. A result of primary interest for us is the local Lipschitz continuity of the gradient of the Moreau envelope of a prox-regular function and the formula $\nabla e_\lambda f(x) = \frac{1}{\lambda}(x - P_\lambda f(x))$. See [17], for a recent analysis of the differential properties of the Moreau envelope in the infinite dimensional setting. Although, the proof of Lipschitz continuity can be found at several places in the literature [31, 33, 17], we did not find a computation of the local Lipschitz constant. We extend the proof in [33] of the Lipschitz continuity, and determine $\lambda^{-1}$ to be the local Lipschitz modulus for small $\lambda$.

The proof of convergence of many methods can be conducted in a general abstract setting. Attouch et al. [2] proved a convergence result for sequences that obey a certain *sufficient decrease condition*, a *relative error condition*, and a *continuity condition*. Under the additional (mild) assumption that the objective function has the so-called Kurdyka–Łojasiewicz (KL) property[1] [19, 25, 26, 4], the length of the sequence is proved to be finite, and the sequence converges to a stationary point of the objective. While this abstract concept can be used to prove global convergence in the non-convex setting of the gradient descent method, forward–backward splitting, and many other algorithms, it seems to be limited to single-step methods. Therefore [29] proved a slightly different result for abstract descent methods, which is applicable to multi-step methods, such as the Heavy-ball method which consider a part of the history of the sequence of iterates. In [28], an abstract convergence result is proved that unifies [2, 13, 29, 27]. The local convergence results of [2] for the setting in [29] is proved in this paper. Under some mild conditions, a sequence that starts in a neighborhood of a local minimum stays within this neighborhood and converges to the minimum. The result can be applied directly to the convergence analysis of the Heavy-ball method and iPiano.

**Outline.** Section 2 introduces the notation, definitions, and basic results that are used in this paper. In Section 3 the conditions for global convergence of abstract descent methods [29, 27] are recapitulated, and the local convergence result for local and global minima is stated. All proofs of the paper are postponed to the appendix. In the subsequent Section 4, important results about the gradient of the Moreau envelope for a prox-regular function are presented. Then, the abstract local convergence results are translated to a statement of iPiano's local convergence in Section 5. Relations to inertial averaged/alternating minimization are discussed in Section 5.2. Applications of these relations are analysed in Section 5.3.

# 2 Preliminaries

Throughout this paper, we will always work in a finite dimensional Euclidean vector space $\mathbb{R}^N$ of dimension $N \in \mathbb{N}$, where $\mathbb{N} := \{1, 2, \ldots\}$. The vector space is equipped with the standard Euclidean norm $|\cdot|$ that is induced by the standard Euclidean inner product $|\cdot| = \sqrt{\langle \cdot, \cdot \rangle}$.

As usual, we consider extended read-valued functions $f : \mathbb{R}^N \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, that are defined on the whole space with *domain* given by $\operatorname{dom} f := \{x \in \mathbb{R}^N \,|\, f(x) < +\infty\}$. A function is called *proper* if it is nowhere $-\infty$ and not everywhere $+\infty$. We define the

[1] For the KL property, we refer to [4, 5, 1, 2].

*epigraph* of the function $f$ as $\operatorname{epi} f := \{(x,\mu) \in \mathbb{R}^{N+1} \,|\, \mu \geq f(x)\}$. The range of a set-valued mapping, which we write as $T\colon \mathbb{R}^N \rightrightarrows \mathbb{R}^M$, is defined as $\operatorname{rge} T := \bigcup_{x\in\mathbb{R}^N} T(x)$.

A key concept in optimization and variational analysis is that of Lipschitz continuity. Sometimes, also the term strict continuity is used, which we define as in [33]:

**Definition 1** (strict continuity)**.** *A single-valued mapping $F\colon D \to \mathbb{R}^M$ defined on $D \subset \mathbb{R}^N$ is* strictly continuous *at $\bar{x}$ if $\bar{x} \in D$ and the value*

$$\operatorname{lip} F(\bar{x}) := \limsup_{\substack{x,x'\to\bar{x} \\ x\neq x'}} \frac{|F(x') - F(x)|}{|x' - x|}$$

*is finite and* $\operatorname{lip} F(\bar{x})$ *is the* Lipschitz modulus *of $F$ at $\bar{x}$. This is the same as saying $F$ is locally Lipschitz continuous at $\bar{x}$ on $D$.*

For convenience, we denote the class of smooth functions whose gradient is strictly continuous by $\mathcal{C}^{1+}$.

The *Fréchet subdifferential* of $f$ at $\bar{x} \in \operatorname{dom} f$ is the set $\widehat{\partial} f(\bar{x})$ of those elements $v \in \mathbb{R}^N$ such that

$$\liminf_{\substack{x\to\bar{x} \\ x\neq\bar{x}}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x}\rangle}{|x - \bar{x}|} \geq 0\,.$$

For $\bar{x} \notin \operatorname{dom} f$, we set $\widehat{\partial} f(\bar{x}) = \emptyset$. For convenience, we introduce *$f$-attentive convergence*: A sequence $(x^n)_{n\in\mathbb{N}}$ is said to *$f$-converge* to $\bar{x}$ if

$$x^n \to \bar{x} \quad \text{and} \quad f(x^n) \to f(\bar{x}) \quad \text{as } n \to \infty\,,$$

and we write $x^n \xrightarrow{f} \bar{x}$. The so-called *(limiting) subdifferential* of $f$ at $\bar{x} \in \operatorname{dom} f$ is defined by

$$\partial f(\bar{x}) := \{v \in \mathbb{R}^N \,|\, \exists\, x^n \xrightarrow{f} \bar{x},\ v^n \in \widehat{\partial} f(x^n),\ v^n \to v\}\,,$$

and $\partial f(\bar{x}) = \emptyset$ for $\bar{x} \notin \operatorname{dom} f$. A point $\bar{x} \in \operatorname{dom} f$ for which $0 \in \partial f(\bar{x})$ is a called a *critical point*. As a direct consequence of the definition of the limiting subdifferential, we have the following closedness property:

$$x^n \xrightarrow{f} \bar{x},\ v^n \to \bar{v},\ \text{and for all } n \in \mathbb{N}\colon v^n \in \partial f(x^n) \quad \Longrightarrow \quad \bar{v} \in \partial f(\bar{x})\,.$$

For a function $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ and $\lambda > 0$, we define the *Moreau envelope*

$$e_\lambda f(x) := \inf_{w\in\mathbb{R}^N} f(w) + \frac{1}{2\lambda}|w - x|^2\,.$$

and the *proximal mapping*

$$P_\lambda f(x) := \arg\min_{w\in\mathbb{R}^N} f(w) + \frac{1}{2\lambda}|w - x|^2\,.$$

For a general function $f$ it might happen that $e_\lambda f(x)$ takes the values $\pm\infty$ and the proximal mapping is empty, i.e. $P_\lambda f(x) = \emptyset$. Therefore, the analysis of the Moreau envelope is usually coupled with the following property.

**Definition 2.** *A function* $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ *is* prox-bounded*, if there exists* $\lambda > 0$ *such that* $e_\lambda f(x) > -\infty$ *for some* $x \in \mathbb{R}^N$*. The supremum of the set of all such* $\lambda$ *is the* threshold $\lambda_f$ *of prox-boundedness for* $f$*.*

A particular interesting (broad) class of functions contains all prox-regular functions. These functions have many favorable properties locally, which otherwise only convex functions reveal.

**Definition 3** (prox-regularity, [33, Def. 13.27])**.** *A function* $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ *is* prox-regular *at* $\bar{x}$ *for* $\bar{v}$ *if* $f$ *is finite and locally lsc at* $\bar{x}$ *with* $\bar{v} \in \partial f(\bar{x})$*, and there exists* $\varepsilon > 0$ *and* $\rho \geq 0$ *such that*

$$f(x') \geq f(x) + \langle v, x' - x\rangle - \frac{\rho}{2}|x' - x|^2 \quad \forall x' \in B_\varepsilon(\bar{x})$$
$$\text{when } v \in \partial f(x),\ |v - \bar{v}| < \varepsilon,\ |x - \bar{x}| < \varepsilon,\ f(x) < f(\bar{x}) + \varepsilon\,.$$

*When this holds for all* $\bar{v} \in \partial f(\bar{x})$*,* $f$ *is said to be prox-regular at* $\bar{x}$*.*

Note that prox-regular functions are not subdifferntially continuous at $\bar{x}$ for $\bar{v}$, i.e. convergence of $(x^\nu, v^\nu) \in \operatorname{Graph}\partial f$ to $(\bar{x}, \bar{v})$ implies convergence of $f(x^\nu) \to f(\bar{x})$ is not satisfied.

For the proof of the Lipschitz property of the Moreau envelope, it will be helpful to consider a so-called localization. A *localization* of $\partial f$ around $(\bar{x}, \bar{v})$ is a mapping $T\colon \mathbb{R}^N \rightrightarrows \mathbb{R}^N$ whose graph is obtained by intersecting $\operatorname{Graph}\partial f$ with some neighborhood of $(\bar{x}, \bar{v})$. We talk about an *$f$-attentive localization* when the above mentioned neighborhood comes from the topology of $f$-attentive convergence in the $x$ component and the ordinary topology on $\mathbb{R}^N$ in the $v$ component.

Finally, the convergence result we build on is only valid for functions that have the KL property at a certain point of interest. This property is shared for example of semi-algebraic functions, globally analytic functions, or, more general, functions definable in an $\mathfrak{o}$-minimal structure. For a details, we refer to [4, 5].

**Definition 4** (Kurdyka–Łojasiewicz property / KL property)**.** *Let* $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ *be an extended real valued function and let* $\bar{x} \in \operatorname{dom}\partial f$*. If there exists* $\eta \in (0, \infty]$*, a neighborhood* $U$ *of* $\bar{x}$ *and a continuous concave function* $\varphi\colon [0, \eta) \to \mathbb{R}_+$ *such that*

$$\varphi(0) = 0, \quad \varphi \in C^1((0, \eta)), \quad \text{and} \quad \varphi'(s) > 0 \text{ for all } s \in (0, \eta),$$

*and for all* $x \in U \cap [f(\bar{x}) < f(x) < f(\bar{x}) + \eta]$ *holds the Kurdyka–Łojasiewicz inequality*

$$\varphi'(f(x) - f(\bar{x}))\|\partial f(x)\|_- \geq 1\,, \tag{1}$$

*then the function has the Kurdyka–Łojasiewicz property at* $\bar{x}$*, where* $\|\partial f(x)\|_- := \inf_{v \in \partial f(x)} |v|$ *is the* lazy slope *(note:* $\inf \emptyset := +\infty$*).*

*If, additionally, the function is lower semi-continuous and the property holds for each point in* $\operatorname{dom}\partial f$*, then* $f$ *is called a Kurdyka–Łojasiewicz function.*

# 3 Abstract Convergence Result for KL Functions

In this section, we establish an abstract local convergence result for descent methods. It is based on a global convergence result proved in [29] for KL functions, which itself is motivated by a slightly different result in [2]. In analogy to [2], we prove a local convergence result of the setting in [29]. *All proofs of this section are in the appendix.*

## 3.1 Global Convergence Results

The convergence result in [29] is based on three abstract conditions for a sequence $(z^k)_{k\in\mathbb{N}} := (x^k, x^{k-1})_{k\in\mathbb{N}}$ in $\mathbb{R}^{2N}$, $x^k \in \mathbb{R}^N$, $x^{-1} \in \mathbb{R}^N$. Fix two positive constants $a > 0$ and $b > 0$ and consider a proper lower semi-continuous function $\mathcal{F}\colon \mathbb{R}^{2N} \to \overline{\mathbb{R}}$. Then, the conditions for $(z^k)_{k\in\mathbb{N}}$ are as follows:

(H1) For each $k \in \mathbb{N}$, it holds that

$$\mathcal{F}(z^{k+1}) + a|x^k - x^{k-1}|^2 \leq \mathcal{F}(z^k)\,.$$

(H2) For each $k \in \mathbb{N}$, there exists $w^{k+1} \in \partial\mathcal{F}(z^{k+1})$ such that

$$|w^{k+1}| \leq \frac{b}{2}(|x^k - x^{k-1}| + |x^{k+1} - x^k|)\,.$$

(H3) There exists a subsequence $(z^{k_j})_{j\in\mathbb{N}}$ such that

$$z^{k_j} \to \tilde{z} \quad \text{and} \quad \mathcal{F}(z^{k_j}) \to \mathcal{F}(\tilde{z})\,, \qquad \text{as } j \to \infty\,.$$

For convenience of the reader, we state the convergence result of [29] here:

**Theorem 5.** *Let $(z^k)_{k\in\mathbb{N}} = (x^k, x^{k-1})_{k\in\mathbb{N}}$ be a sequence that satisfies (H1), (H2), and (H3) for a proper lower semi-continuous function $\mathcal{F}\colon \mathbb{R}^{2N} \to \overline{\mathbb{R}}$ which has the KL property at the cluster point $\tilde{z}$ specified in (H3).*
*Then, the sequence $(x^k)_{k\in\mathbb{N}}$ has finite length, i.e.*

$$\sum_{k=1}^{\infty} |x^k - x^{k-1}| < +\infty\,,$$

*and converges to $\bar{z} = \tilde{z}$ where $\bar{z} = (\bar{x}, \bar{x})$ is a critical point of $\mathcal{F}$.*

*Remark* 1. In view of the proof of this statement, it is clear that the same result can be established when (H1) is replaced by $\mathcal{F}(z^{k+1}) + a|x^{k+1} - x^k|^2 \leq \mathcal{F}(z^k)\,.$

## 3.2 Local Convergence Results

The local convergence result shows that, once entered a region of attraction (around a local minimum), all iterates of a sequence $(z^k)_{k\in\mathbb{N}}$ satisfying (H1), (H2) and a certain growth condition (H4) stay in a neighborhood of this minimum and converge to the minimum. As a rather obvious consequence, this result also applies to a global minimum. However, we establish local convergence to a global minimum without the need of the growth condition (H4). This result can be used to prove local convergence of an abstract descent method for feasibility problems.

In the following, for $z \in \mathbb{R}^{2N}$ we denote by $z_1, z_2 \in \mathbb{R}^N$ the first and second block of coordinates $z = (z_1, z_2)$. The same holds for other vectors in $\mathbb{R}^{2N}$. The growth condition just mentioned is:

(H4) For any $\delta > 0$ there exist $0 < \rho < \delta$ and $\nu > 0$ such that

$$z \in B_\rho(z^*)\,,\ \mathcal{F}(z) < \mathcal{F}(z^*) + \nu\,,\ y_2 \notin B_\delta(z_2^*)\ \Rightarrow\ \mathcal{F}(z) < \mathcal{F}(y) + \frac{a}{4}|z_2 - y_2|^2\,.$$

Under this condition, the following theorem establishes the local convergence result. Its formulation is adjusted to the corresponding one in [2].

**Theorem 6.** *Let $\mathcal{F}\colon \mathbb{R}^{2N} \to \overline{\mathbb{R}}$ be a proper lower semi-continuous function which has the KL property at some local minimizer $z^* = (x^*, x^*)$ of $\mathcal{F}$. Assume (H4) holds at $z^*$.*
*Then, for any $r > 0$, there exist $u \in (0, r)$ and $\mu > 0$ such that the conditions*

$$z^0 \in B_u(z^*)\,, \qquad \mathcal{F}(z^*) < \mathcal{F}(z^0) < \mathcal{F}(z^*) + \mu\,, \tag{2}$$

*imply that any sequence $(z^k)_{k\in\mathbb{N}}$ that starts at $z^0$ and satisfies (H1) and (H2) has the finite length property and remains in $B_r(z^*)$ and converges to some $\bar z \in B_r(z^*)$, a critical point of $\mathcal{F}$ with $\mathcal{F}(\bar z) = \mathcal{F}(z^*)$.*

Again in analogy to [2], we verify a simple condition that implies (H4).

**Lemma 7.** *Let $\mathcal{F}\colon \mathbb{R}^{2N} \to \overline{\mathbb{R}}$ be a proper lower semi-continuous function and $z^* = (x^*, x^*) \in \operatorname{dom} \mathcal{F}$ a local minimum. Suppose, for any $\delta > 0$, $\mathcal{F}$ satisfies the growth condition*

$$\mathcal{F}(y) \geq \mathcal{F}(z^*) - \frac{a}{16}|y_2 - z_2^*|^2 \quad \forall y \in \mathbb{R}^{2N}, y_2 \notin B_\delta(z_2^*)\,.$$

*Then, $\mathcal{F}$ satisfies (H4).*

The following theorem and corollary are immediate consequences of Theorem 6 and Lemma 7 and their proof is obvious.

**Theorem 8.** *Let $\mathcal{F}\colon \mathbb{R}^{2N} \to \overline{\mathbb{R}}$ be a proper lower semi-continuous function which has the KL property at a global minimizer $z^* = (x^*, x^*)$ of $\mathcal{F}$.*
*Then, for any $r > 0$, there exist $u \in (0, r)$ and $\mu > 0$ such that the conditions*

$$z^0 \in B_u(z^*)\,, \qquad \mathcal{F}(z^*) < \mathcal{F}(z^0) < \mathcal{F}(z^*) + \mu\,, \tag{3}$$

*imply that any sequence* $(z^k)_{k\in\mathbb{N}}$ *that starts at* $z^0$ *and satisfies (H1) and (H2) has the finite length property and remains in* $B_r(z^*)$ *and converges to a global minimizer* $\bar z \in B_r(z^*)$*, i.e.* $\mathcal{F}(\bar z) = \min \mathcal{F}$*.*

*Remark* 2. The assumption in (H4) and Lemma 7 only restrict the behavior of the function along the second block of coordinates of $z = (z_1, z_2) \in \mathbb{R}^{N\times 2}$. This makes sense, because, for sequences that we consider, the first and second block are dependent from each other.

**Corollary 9.** *Let* $S_1, \dots, S_M \subset \mathbb{R}^N$ *be semi-algebraic sets such that* $\bigcap_{i=1}^M S_i \neq \emptyset$ *and let* $F\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ *be given by* $F(x) = \frac{1}{2}\sum_{i=1}^M \mathrm{dist}(x, S_i)^2$*. For a constant* $c \geq 0$*, we consider the objective function*

$$\mathcal{F}(z) = \mathcal{F}(z_1, z_2) = F(z_1) + c|z_1 - z_2|^2 \,.$$

*Suppose* $z^* = (x^*, x^*)$ *is a global minimizer of* $\mathcal{F}$*, i.e.,* $x^* \in \bigcap_{i=1}^M S_i$*. Then, for* $z^0 = (x^0, x^{-1})$ *sufficiently close to* $z^*$*, any algorithm that satisfies (H1) and (H2) and starts at* $z^0$ *generates a sequence that*

- *remains in a neighborhood of* $z^*$*,*
- *has the finite length property,*
- *and converges to a point* $\bar z = (\bar x, \bar x)$ *with* $\bar x \in \bigcap_{i=1}^M S_i$*.*

# 4 The Gradient of the Moreau Envelope

It is well known that for a proper lower semi-continuous convex function $f$ the Moreau envelope is defined everywhere. Moreover, it is known to be differentiable with a closed form expression of the gradient

$$\nabla e_\lambda f(x) = \frac{1}{\lambda}(x - P_\lambda f(x))\,, \tag{4}$$

which can be shown to be Lipschitz continuous with constant $\lambda^{-1}$.

In the following, we discuss an extension of this statement to the non-convex setting, for prox-regular functions. It turns out that a similar statement is valid locally. The third item of the following proposition extends [33, Prop. 13.37] by an estimation of the local Lipschitz constant of the gradient of the Moreau envelope. In order to prove Item (iii), we develop the basic objects that are required in the same way as [33, Prop. 13.37]. Thus, the first part of the proof (see appendix) coincides with [33, Prop. 13.37].

**Proposition 10.** *Suppose that* $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ *is prox-regular at* $\bar x$ *for* $\bar v = 0$*, and that* $f$ *is prox-bounded. Then for all* $\lambda > 0$ *sufficiently small there is a neighborhood of* $\bar x$ *on which*

*(i)* $P_\lambda f$ *is monotone, single-valued and Lipschitz continuous and* $P_\lambda f(\bar x) = \bar x$*.*

*(ii)* $e_\lambda f$ *is differentiable with* $\nabla(e_\lambda f)(\bar{x}) = 0$*, in fact of class* $\mathcal{C}^{1+}$ *with*

$$\nabla e_\lambda f = \lambda^{-1}(\mathrm{id} - P_\lambda f) = (\lambda I + T^{-1})^{-1}$$

*for an* $f$*-attentive localization* $T$ *of* $\partial f$ *at* $(\bar{x}, 0)$*. Indeed, this localization can be chosen so that the set* $U_\lambda := \mathrm{rge}\,(I+\lambda T)$ *serves for all* $\lambda > 0$ *sufficiently small as a neighborhood of* $\bar{x}$ *on which these properties hold.*

*(iii) There is a neighborhood of* $\bar{x}$ *on which for small* $\lambda$ *enough the local Lipschitz constant of* $\nabla e_\lambda f$ *is* $\lambda^{-1}$*. If we denote by* $\lambda_0$ *the modulus in the subgradient inequality of the prox-regularity,* $\lambda$ *must satisfy*

$$0 < \lambda \leq \frac{\lambda_0}{2}\,.$$

The third part of the proof is motivated by a similar derivation for distance function and projection operators in [21].

# 5 Inertial Averaged/Alternating Proximal Minimization

The application of gradient descent, forward-backward splitting and inertial variants to a special setting in the objective function recovers relations to the averaged proximal minimization (resp. projection) and the alternating proximal minimization (resp. projection) method. Therefore, convergence results translate directly. In this section, we review the convergence results of the inertial forward–backward method called iPiano [29, 27] (see also [7]) and state a local convergence result, which follows from Theorem 6. The local convergence allows us to invoke the result of Section 4 for prox-regular and prox-bounded functions along the iterations of iPiano, which is key for the mentioned connections.

## 5.1 Heavy-ball Method and iPiano

iPiano applies to structured nonsmooth and nonconvex optimization problems with a proper lower semi-continuous extended valued function $h\colon \mathbb{R}^N \to \overline{\mathbb{R}}$, $N \geq 1$:

$$\min_{x\in\mathbb{R}^N} h(x)\,, \qquad h(x) = f(x) + g(x)\,. \tag{5}$$

The function $f\colon \mathbb{R}^N \to \mathbb{R}$ is assumed to be $C^1$-smooth (possibly nonconvex) with $L$-Lipschitz continuous gradient on $\mathrm{dom}\, g$, $L > 0$. Further, let the function $g\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ be simple (possibly nonsmooth and nonconvex) and prox-bounded. Simple refers to the fact that the associated proximal map can be solved efficiently for the global optimum. Furthermore, we require $h$ to be coercive and bounded from below by some value $\underline{h} > -\infty$. The coercivity property could be replaced by the assumption that the sequence that is generated by the algorithm is bounded.

**Algorithm 1.** *iPiano*

- ***Initialization***: *Choose a starting point* $x^0 \in \operatorname{dom} h$ *and set* $x^{-1} = x^0$.
- ***Iterations*** $(k \geq 0)$: *Update:*

$$
\begin{aligned}
y^k &= x^k + \beta(x^k - x^{k-1}) \\
x^{k+1} &\in \arg\min_{x \in \mathbb{R}^N} \; g(x) + \left\langle \nabla f(x^k), x - x^k \right\rangle + \frac{1}{2\alpha}|x - y^k|^2 \,.
\end{aligned}
\tag{6}
$$

- ***Parameter setting***: *See Table 1.*

The algorithm iPiano is outlined in Algorithm 1. The Heavy-ball method is recognized as a special case of iPiano where the non-smooth function $g = 0$. It is sometimes called inertial gradient descent method or gradient descent with momentum.

*Remark* 3. For simplicity, we consider only the constant step size version of iPiano. For a variable step size version including a backtracking procedure, we refer to [29, 27], and to [28] for a variable metric and block coordinate descent version of iPiano.

Considering local convergence, the properties of (5) are required to hold only on a neighborhood of a local minimum. Thanks to the local convergence result in Theorem 6, starting close enough to a local or global minimum, all iterates stay within this neighborhood.

In [27], functions $g$ that are semi-convex received special attention. The resulting step size restrictions for semi-convex functions $g$ are similar to those of convex functions. A function is said to be semi-convex with modulus $m \in \mathbb{R}$, if $m$ is the largest value such that $g(x) - \frac{m}{2}|x|^2$ is convex. For convex functions, $m = 0$ holds, and for strongly convex functions $m > 0$. We assume $m < L$. According to [33, Theorem 10.33], saying a function $g$ is (locally) semi-convex on an open set $V \subset \operatorname{dom} g$ is the same as saying $g$ is lower-$\mathcal{C}^2$ on $V$.

The following theorem is the convergence theorem in [27] and states the global convergence to a stationary point. The subsequent theorem states the local convergence, which in general also requires (H4)to be satisfied.

**Theorem 11.** *Let* $(x^k)_{k\in\mathbb{N}}$ *be generated by Algorithm 1. Then, the sequence* $(z^k)_{k\in\mathbb{N}}$ *with* $z^k = (x^k, x^{k-1})$ *satisfies (H1), (H2), (H3) for the function*

$$
H_\delta \colon \mathbb{R}^{2N} \to \mathbb{R} \cup \{\infty\}\,, \quad (x, y) \mapsto h(x) + \delta|x - y|^2 \,.
$$

*Moreover, if* $H_\delta(x, y)$ *has the Kurdyka–Łojasiewicz property at a cluster point* $z^* = (x^*, x^*)$, *then the sequence* $(x^k)_{k\in\mathbb{N}}$ *has the finite length property,* $x^k \to x^*$ *as* $k \to \infty$, *and* $z^*$ *is a critical point of* $H_\delta$, *hence* $x^*$ *is a critical point of* $h$.

**Theorem 12.** *Let* $(x^n)_{n\in\mathbb{N}}$ *be generated by Algorithm 1. If* $x^*$ *is a local (or global) minimizer of* $h$, *then* $z^* = (x^*, x^*)$ *is a local (or global) minimizer of* $H_\delta$ *(defined in Theorem 11). Suppose (H4) holds at* $z^*$ *and* $H_\delta$ *has the KL property at* $z^*$.

| Method | $f$ | $g$ | $\alpha$ | $\beta$ |
|---|---|---|---|---|
| Gradient Descent | $f \in \mathcal{C}^{1+}$ | $g \equiv 0$ | $\alpha \in (0, \frac{2}{L})$ | $\beta = 0$ |
| Heavy-ball method | $f \in \mathcal{C}^{1+}$ | $g \equiv 0$ | $\alpha \in (0, \frac{2(1-\beta)}{L})$ | $\beta \in [0, 1)$ |
| PPA | $f \equiv 0$ | $g$ convex | $\alpha > 0$ | $\beta = 0$ |
| FBS | $f \in \mathcal{C}^{1+}$ | $g$ convex | $\alpha \in (0, \frac{2}{L})$ | $\beta = 0$ |
| FBS (non-convex) | $f \in \mathcal{C}^{1+}$ | $g$ non-convex | $\alpha \in (0, \frac{1}{L})$ | $\beta = 0$ |
| iPiano | $f \in \mathcal{C}^{1+}$ | $g$ convex | $\alpha \in (0, \frac{2(1-\beta)}{L})$ | $\beta \in [0, 1)$ |
| iPiano | $f \in \mathcal{C}^{1+}$ | $g$ non-convex | $\alpha \in (0, \frac{(1-2\beta)}{L})$ | $\beta \in [0, \frac{1}{2})$ |
| iPiano | $f \in \mathcal{C}^{1+}$ | $g$ $m$-semi-convex | $\alpha \in (0, \frac{2(1-\beta)}{L-m})$ | $\beta \in [0, 1)$ |

Table 1: Convergence of iPiano as stated in Theorems 11 and 12 is guaranteed for the parameter settings listed in this table. Note that for local convergence, also the required properties of $f$ and $g$ are required to hold only locally. iPiano has several well-known special cases, such as the gradient descent method, Heavy-ball method, proximal point algorithm (PPA), and forward–backward splitting (FBS).

*If $z^0$ is sufficiently close to $z^*$, such that $z^0 \in B_u(z^*)$ and $\mathcal{F}(z^*) < \mathcal{F}(z^0) < \mathcal{F}(z^*) + \mu$ holds for some $u, \mu > 0$, then the sequence $(x^n)_{n\in\mathbb{N}}$ has finite length, the sequence $z^n$ remains in $B_r(z^*)$, and $x^n \xrightarrow{h} x^*$ as $n \to \infty$.*

## 5.2 A Special Non-convex Setting of iPiano

Throughout the whole section, we assume that the gradient of the Moreau envelope can be expressed as in (4) along the sequence of iterates. In Section 4, we already discussed situation when this is happening. This can be true globally or on a neighborhood of a local (or global) minimum. If this property can only be guaranteed locally, under (H4), we know that all iterates stay withing this neighborhood. Note that proximal mappings that are derived via (4) are necessarily single-valued whereas the proximal mapping appearing in (6) may be multi-valued.

**Heavy-ball method on the Moreau envelope.** The Heavy-ball method can be applied to

$$\min_{x\in\mathbb{R}^N} F(x) , \qquad F(x) = e_\lambda f(x) = \min_{w\in\mathbb{R}^N} f(w) + \frac{1}{2\lambda}|w - x|^2 .$$

The Lipschitz constant of $\nabla e_\lambda f(x)$ is $\lambda^{-1}$. The step size restriction inferred from Table 1 is $0 < \alpha < 2(1 - \beta)\lambda$ and $\beta \in [0, 1)$. The algorithm's update reads, using $\theta := \alpha\lambda^{-1}$ and $y^k = x^k + \beta(x^k - x^{k-1})$ as follows:

$$\begin{aligned} x^{k+1} &= y^k - \alpha \nabla e_\lambda f(x^k) \\ &= x^k - \alpha\lambda^{-1}(x^k - P_\lambda f(x^k)) + \beta(x^k - x^{k-1}) \\ &= (1 - \theta)x^k + \theta P_\lambda f(x^k) + \beta(x^k - x^{k-1}) . \end{aligned} \tag{7}$$

The iteration step is the inertial *proximal point algorithm* for $\theta = 1$, which is feasible for $\beta \in [0, \frac{1}{2})$ and $\alpha = \lambda$. For a prox-regular function, the algorithm converges locally, whereas for a convex function $f$, globally convergence is guaranteed.

**Heavy-ball method on the sum of two Moreau envelopes.** Of course, the Heavy-ball method can be applied to the sum of Moreau envelope functions

$$\begin{aligned} F(x) &= \frac{1}{2}\left(e_\lambda g(x) + e_\lambda f(x)\right) \\ &= \min_{w,z\in\mathbb{R}^N} \frac{1}{2}\left(g(z) + f(w) + \frac{1}{2\lambda}|z-x|^2 + \frac{1}{2\lambda}|w-x|^2\right). \end{aligned}$$

The Lipschitz constant of $\nabla F$ is $\lambda^{-1}$. In analogy to the preceding consideration, we obtain

$$\begin{aligned} x^{k+1} &= y^k - \frac{\alpha}{2}\left(\nabla e_\lambda g(x^k) + \nabla e_\lambda f(x^k)\right) \\ &= \frac{1}{2}\left(y^k - \alpha\nabla e_\lambda g(x^k)\right) + \frac{1}{2}\left(y^k - \alpha\nabla e_\lambda f(x^k)\right) \\ &= \frac{1}{2}\left((1-\theta)x^k + \theta P_\lambda g(x^k)\right) + \frac{1}{2}\left((1-\theta)x^k + \theta P_\lambda f(x^k)\right) + \beta(x^k - x^{k-1}) \\ &= (1-\theta)x^k + \frac{\theta}{2}\left(P_\lambda g(x^k) + P_\lambda f(x^k)\right) + \beta(x^k - x^{k-1}) \end{aligned}$$

If $\theta = 1$ is feasible, we obtain the *averaged proximal minimization method* (respectively, *averaged projection method* if $f$ and $g$ are indicator functions). This scheme has an obvious extension to the weighted average of a finite sum of Moreau envelopes. For a prox-regular functions $f$ and $g$, the algorithm converges locally, whereas for a convex functions $f$ and $g$, globally convergence is guaranteed..

**iPiano on an objective involving a Moreau envelope.** Let us now consider the minimization problem

$$\min_{x\in\mathbb{R}^N} g(x) + F(x), \qquad F(x) = e_\lambda f(x) = \min_{w\in\mathbb{R}^N} f(w) + \frac{1}{2\lambda}|w-x|^2.$$

The Lipschitz constant of $\nabla F$ is $\lambda^{-1}$ and iPiano is feasible either for $m$-semi-convex $g$ with $\alpha < 2(1-\beta)/(\lambda^{-1} - m)$ and $\beta \in [0, 1)$ or for general $g$ with $\alpha < (1-2\beta)\lambda$ and $\beta \in [0, \frac{1}{2})$. iPiano can be written as follows:

$$\begin{aligned} x^{k+1} &= P_\alpha g(y^k - \alpha\nabla e_\lambda f(x^k)) \\ &= P_\alpha g((1-\theta)x^k - \theta P_\lambda f(x^k) + \beta(x^k - x^{k-1})) \end{aligned}$$

The resulting update scheme simplifies for $\theta = 1$ to the *alternating proximal minimization method* (respectively, *alternating projection method* if $f$ and $g$ are indicator functions). For a prox-regular function $f$, the algorithm converges locally, whereas for a convex function $f$, globally convergence is guaranteed. Note that the global convergence result allows for non-convex functions $g$.

**iPiano for finite sums of non-convex simple functions.** Consider the minimization probelm

$$\min_{x\in\mathbb{R}^N} \sum_{i=1}^M g_i(x)$$

with possibly non-convex functions $g_i$, or its equivalent form

$$\min_{x_1,\ldots,x_M\in\mathbb{R}^N} \sum_{i=1}^M g_i(x_i)\,, \quad s.t.\ x_1 = \ldots = x_M\,.$$

In order to solve this problem, we relax the constraint and consider the minimization problem

$$\min_{x_1,\ldots,x_M\in\mathbb{R}^N} \sum_{i=1}^M g_i(x_i) + e_\lambda\delta_C(x_1,\ldots,x_M)\,,$$

where $C = \{x_1,\ldots,x_M \in \mathbb{R}^N \,|\, x_1 = \ldots = x_M\}$ denotes the convex constraint set. The Moreau envelope of the indicator function measures the distance of $(x_1,\ldots,x_M)$ to the convex feasible set $C$. It is continuously differentiable with 1-Lipschitz continuous gradient. Applying iPiano to this optimization problem allows for general non-convex funcitons $g_1,\ldots,g_M$ with simple proximal mappings. Note that the proximal mappings can be computed for each function $g_i$ independently. iPiano reads as follows:

$$\begin{aligned} x_i^{k+1} = P_\theta g_i(\bar x_i^{k+1})\,, \quad \bar x_i^{k+1} &= x_i^k - \theta(x_i^k - \mathrm{proj}_C(x_1^k,\ldots,x_M^k)) + \beta(x_i^k - x_i^{k-1}) \\ &= x_i^k - \frac{\theta}{M}\sum_{j=1}^M x_j^k + \beta(x_i^k - x_i^{k-1})\,. \end{aligned}$$

In the most general setting, where $g_1,\ldots,g_M$ are non-convex simple functions, the feasible step size is $\theta \in (0,1)$ with $\beta \in [0,\frac{1}{2})$, and global convergence is guaranteed. Unfortunately, we solve the relaxed problem, which minimizes the distance to the feasible set. The solution of this iterative procedure cannot be guaranteed to be a feasible solution of the original problem. Projecting the solution of the relaxed problem to the feasible set might give a good approximation of the original problem.

## 5.3 Applications

### 5.3.1 A Feasibility Problem

We consider the example in [20] that demonstrates (local) linear convergence of the alternating projection method. In this experiment, we experimentally verify the linear convergence of the inertial alternating projection and the inertial averaged projection method. Inertial methods clearly outperform the corresponding standard methods.

The goal is to find an $N\times M$ matrix $X$ of rank $R$ that satisfies a linear system of equations $\mathcal{A}(X) = b$, i.e.,

$$\text{find} \quad X \quad \text{in} \quad \underbrace{\{X\in\mathbb{R}^{N\times M} \,|\, \mathcal{A}(X) = b\}}_{=:\mathscr{A}} \cap \underbrace{\{X\in\mathbb{R}^{N\times M} \,|\, \mathrm{rank}(X) = R\}}_{=:\mathscr{R}},$$

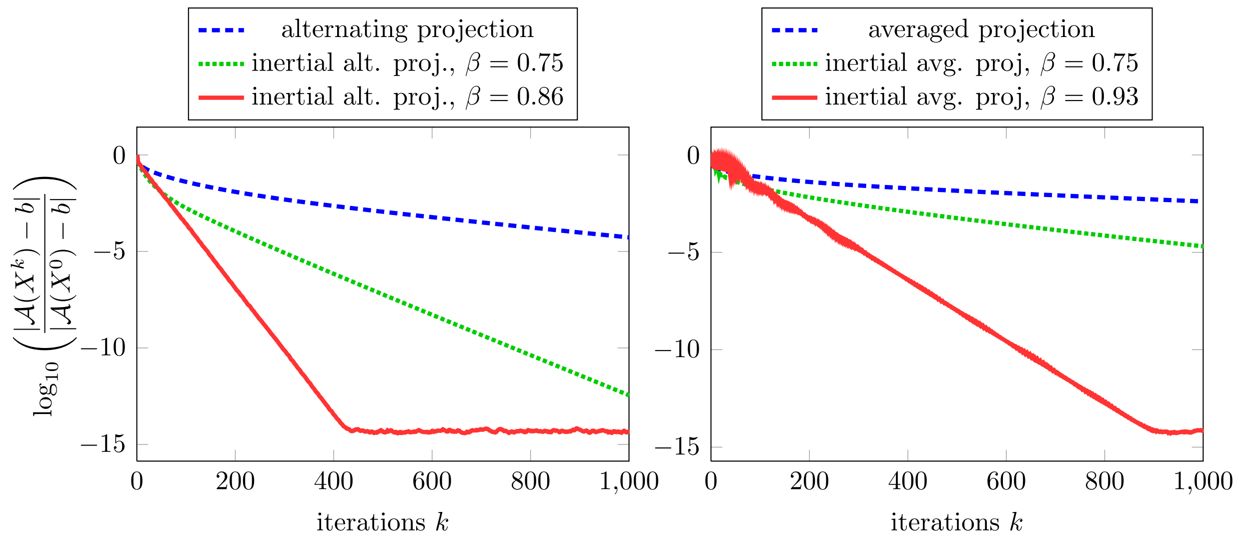


Figure 1: Convergence plots for the feasibility problem in Section 5.3.1. Inertial methods consistently outperform the basic models.

where $\mathcal{A}\colon \mathbb{R}^{N\times M} \to \mathbb{R}^D$ is a linear mapping and $b \in \mathbb{R}^D$. Such feasibility problems are well suited for split projection methods, as the projection onto each set is easy to conduct. The projection onto constraint $\mathscr{A}$ is given by

$$\operatorname{proj}_{\mathscr{A}}(X) = X - \mathcal{A}^*(\mathcal{A}\mathcal{A}^*)^{-1}(\mathcal{A}(X) - b) \quad \text{and} \quad \operatorname{proj}_{\mathscr{R}}(X) = \sum_{i=1}^{R} \sigma_i u_i v_i^\top ,$$

where $USV^\top$ is the singular value decomposition of $X$ with $U = (u_1, u_2, \dots, u_N)$, $V = (v_1, v_2, \dots, v_M)$ and singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N$ sorted in decreasing order along the diagonal of $S$. As in [20], we randomly generate operators $\mathcal{A}$ by constructing random matrices $\mathcal{A}(X) = (\langle A_1, X\rangle, \dots, \langle A_D, X\rangle)$, selecting $b$ such that $\mathcal{A}(X) = b$ has a rank $R$ solution, and the dimensions are chosen as $M \leq N$, $R = 10$, and $MR < D \leq R(N + M - R)$. In all experiments, we observed linear convergence for all methods. In the following, we pick one random problem with $M = 110$, $N = 100$, $R = 4$, $D = 450$, and detail on the convergence of $|\mathcal{A}(X) - B|$. We consider the alternating projection method

$$X^{k+1} = \operatorname{proj}_{\mathscr{A}}(\operatorname{proj}_{\mathscr{R}}(X^k)) ,$$

the averaged projection method

$$X^{k+1} = \frac{1}{2}\left(\operatorname{proj}_{\mathscr{A}}(X^k) + \operatorname{proj}_{\mathscr{R}}(X^k)\right) ,$$

and their inertial variants proposed in Section 5.2, which arise from iPiano. As the inertial variants can be seen as generalizations—they include the additional inertial parameter $\beta$—they can be tuned to perform at least as well as the basic methods for $\beta = 0$. However, it turns out that the inertial term effects the convergence in a positive way. Already using the

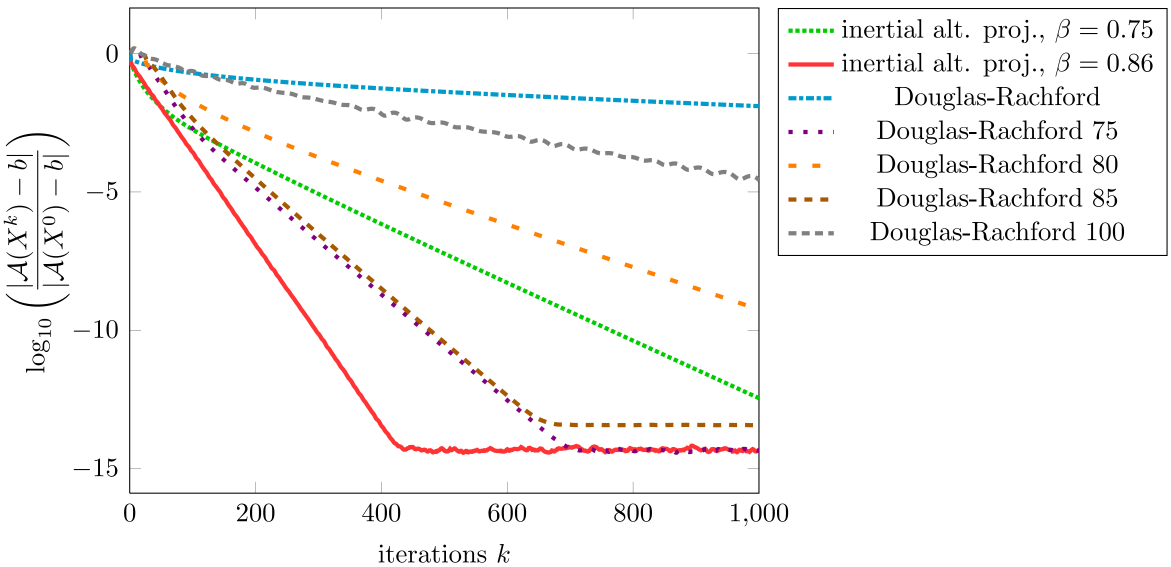


Figure 2: Convergence plots for the feasibility problem in Section 5.3.1. Even after tuning the heuristic step size strategy of the Douglas–Rachford method [22], the proposed inertial alternating projection method performs significantly better.

standard parameter $\beta = 0.75$, inertial methods outperform the basic methods substantially. We call it "standard parameter" as practical experience has shown that it works often well for iPiano in general. Figure 1 shows the convergence plots. The best convergence for the inertial alternating projection method was obtained by setting $\beta = 0.86$ and for the inertial averaged projection method for $\beta = 0.93$. These optimal values are tuned manually.

We also compare our method against the recently proposed globally convergent Douglas–Rachford splitting for non-convex feasibility problems [22]. We incorporated several results for this method in Figure 2. The algorithm depends on a parameter $\gamma$, which in theory is required to be rather small: $\gamma_0 := \sqrt{3/2} - 1$. The basic model "Douglas–Rachford" uses the maximal feasible value for this $\gamma$-parameter. The other results are based on the heuristic proposed in [22], which at least guarantees boundedness of the iterates. We set $\gamma = 150\gamma_0$ and update $\gamma$ by $\max(\gamma/2, 0.9999\gamma_0)$ if $\|y^k - y^{k-1}\| > t/k$. We refer to [22] for the meaning of $y^k$. Since the proposed value $t = 1000$ did not work well in our experiment, we optimized $t$ manually. The values in the legend of Figure 2 refer to the values $t$ that we used. Even after tuning this parameter the inertial alternating projection method converged significantly faster.

### 5.3.2 Examples of Objective Functions Involving Moreau Envelopes

This section is motivated from image processing applications. A standard problem is that of denoising a two-dimensional noisy image represented by a vector $\mathbf{f} \in \mathbb{R}^N$. This task can be solved by minimizing an objective function of type

$$\min_{u\in\mathbb{R}^N} \frac{\lambda}{2}|u-\mathbf{f}|^2 + \mathcal{R}(u)\,, \tag{8}$$

where the first term favors the solution $u$ to stay close to the noisy image $\mathbf{f}$ and the regularizer $\mathcal{R}$ penalizes violations of prior assumptions for $u$. The parameter $\lambda > 0$ steers the importance of both terms. A prominent choice of the regularizer is the convex total variation [34]. However, non-convex regularizers are known to reflect better the natural statistics of images [16]. See [15, 3] for early applications of this paradigm.

The models that we consider here are motivated by ideas from [9] in the convex setting and from [18], where some non-convex problems are considered. Consider the minimization problem (8) with a non-convex regularizer that can be split as $\mathcal{R} = \mathcal{R}^h + \mathcal{R}^v$, into a "horizontal" and a "vertical" component. This includes truncated (anisotropic) total variation or truncated quadratic regularization terms.

There are at least three Lagrangian dual objectives, which are based on different splittings:

$$\begin{aligned}
\mathcal{D}_1(\mu) =&\ \min_{u,v} \frac{\lambda}{4}|u-\mathbf{f}|^2 + \mathcal{R}^h(u) + \frac{\lambda}{4}|v-\mathbf{f}|^2 + \mathcal{R}^v(v) + \langle \mu, u-v\rangle \\
\mathcal{D}_2(\mu) =&\ \min_{u,v} \frac{\lambda}{2}|u-\mathbf{f}|^2 + \mathcal{R}^h(u) + \mathcal{R}^v(v) + \langle \mu, u-v\rangle \\
\mathcal{D}_3(\mu_1,\mu_2) =&\ \min_{u,v,w} \frac{\lambda}{2}|w-\mathbf{f}|^2 + \mathcal{R}^h(u) + \langle \mu_1, w-u\rangle + \mathcal{R}^v(v) + \langle \mu_2, w-v\rangle
\end{aligned}$$

For convex regularizers the last splitting was used in [9]. Given a solution of the dual problem, the primal variables can be recovered for the first and second problem by $v = u = \arg\min_u \mathcal{R}^h(u) + \langle \mu, u\rangle + \frac{\lambda}{4}|u-\mathbf{f}|^2$ and for the last problem by $v = u = w = \mathbf{f} - \frac{\mu_1+\mu_2}{\lambda}$. Unless the objective function is convex, we cannot expect to solve the primal problem (8) exactly. A thorough study of the guarantees in the non-convex setting and the type of regularizers that yield strong duality results is part of future work. Here, we assume that solving the dual problems recovers good primal solutions.

The function $\mathcal{D}_1$ can be recognized as, essentially, the sum of two Moreau envelopes

$$\begin{aligned}
\mathcal{D}_1(\mu) =&\ \left(\min_u \mathcal{R}^h(u) + \frac{\lambda}{4}|\mathbf{f} - \frac{2\mu}{\lambda} - u|^2\right) \\
&+ \left(\min_v \mathcal{R}^v(v) + \frac{\lambda}{4}|\mathbf{f} + \frac{2\mu}{\lambda} - v|^2\right) - \frac{2}{\lambda}|\mu|^2 \\
=&\ e_{2/\lambda}\mathcal{R}^h(\mathbf{f} - 2\mu/\lambda) + e_{2/\lambda}\mathcal{R}^v(\mathbf{f} + 2\mu/\lambda) - \frac{2}{\lambda}|\mu|^2
\end{aligned}$$

with 2-Lipschitz continuous gradient

$$\nabla\mathcal{D}_1(\mu) = P_{2/\lambda}\mathcal{R}^h(f - 2\mu/\lambda) - P_{2/\lambda}\mathcal{R}^v(f + 2\mu/\lambda)\,.$$

Therefore, gradient descent or the heavy ball method from Section 5.2 can be used for maximizing the dual problem.

The second Lagrangian dual can be solved using forward–backward splitting type methods and their respective inertial variants. The objective can be written as

$$\mathcal{D}_2(\mu) = \underbrace{e_{1/\lambda}\mathcal{R}^h(f - \mu/\lambda) - \frac{1}{2\lambda}|\mu|^2}_{\mathcal{D}_2^{\mathrm{fwd}}(\mu)} + \underbrace{\Big(\min_v \mathcal{R}^v(v) - \langle \mu, v - f\rangle\Big)}_{\mathcal{D}_2^{\mathrm{bwd}}(\mu)},$$

where $\mathcal{D}_2^{\mathrm{fwd}}(\mu)$ is the part for which we use the forward step (gradient step) and the backward step (proximal step) is applied to $\mathcal{D}_2^{\mathrm{bwd}}(\mu)$. If we assumed convexity, we could replace $\mathcal{D}_2^{\mathrm{bwd}}$ by the convex conjugate and solve the associate proximal mapping using Moreau's identity. The 1-Lipschitz continuous gradient, which is required for the forward step, is

$$\nabla \mathcal{D}_2^{\mathrm{fwd}}(\mu) = P_{1/\lambda}\mathcal{R}^h(\mathbf{f} - \mu/\lambda) - (\mathbf{f} - \mu/\lambda) - \frac{\mu}{\lambda} = P_{1/\lambda}\mathcal{R}^h(\mathbf{f} - \mu/\lambda) - \mathbf{f}\,.$$

The backward step requires to find a solution $\hat{v}$ and $\hat{\mu}$ of

$$\min_\mu -\mathcal{D}_2^{\mathrm{bwd}}(\mu) + \frac{1}{2\alpha}|\mu - \bar{\mu}|^2\,,$$

where $\bar{\mu}$ is the point after the forward step, for some step size $\alpha$. We compute it as follows. Issues of swapping min and max are postponed to future work.

$$\begin{aligned}
&\min_\mu -\mathcal{D}_2^{\mathrm{bwd}}(\mu) + \frac{1}{2\alpha}|\mu - \bar{\mu}|^2 \\
&= \min_\mu \max_v\; -\mathcal{R}^v(v) + \langle \mu, v - \mathbf{f}\rangle + \frac{1}{2\alpha}|\mu - \bar{\mu}|^2 \\
&= \max_v\; -\mathcal{R}^v(v) - \frac{\alpha}{2}|v|^2 + \langle v, \bar{\mu} + \alpha\mathbf{f}\rangle + \min_\mu\; \frac{1}{2\alpha}|\mu - \bar{\mu} - \alpha(\mathbf{f} - v)|^2 \\
&= \max_v\; -\mathcal{R}^v(v) - \frac{\alpha}{2}|v - (\mathbf{f} + \bar{\mu}/\alpha)|^2 + \min_\mu\; \frac{1}{2\alpha}|\mu - \bar{\mu} - \alpha(\mathbf{f} - v)|^2 \\
&= -\min_v\; \mathcal{R}^v(v) + \frac{\alpha}{2}|v - (\mathbf{f} + \bar{\mu}/\alpha)|^2 - \min_\mu\; \frac{1}{2\alpha}|\mu - \bar{\mu} - \alpha(\mathbf{f} - v)|^2
\end{aligned}$$

From the last line, we can directly infer that

$$\hat{\mu} = \bar{\mu} + \alpha(\mathbf{f} - \hat{v}) = \alpha((\mathbf{f} + \bar{\mu}/\alpha) - \hat{v}) \quad \text{with} \quad \hat{v} = P_{1/\alpha}\mathcal{R}^v(f + \bar{\mu}/\alpha)\,.$$

The third Lagrangian dual function can be rewritten as follows:

$$\begin{aligned}\mathcal{D}_3(\mu_1,\mu_2) = &\min_w \frac{\lambda}{2}|w-\mathbf{f}|^2 + \langle \mu_1+\mu_2, w\rangle \\ &- \max_u \langle \mu_1, u\rangle - \mathcal{R}^h(u) - \max_v \langle \mu_2, v\rangle - \mathcal{R}^v(v) \\ = &\underbrace{\langle \mu_1+\mu_2, \mathbf{f}\rangle - \frac{1}{2\lambda}|\mu_1+\mu_2|^2}_{\mathcal{D}_3^{\text{fwd}}(\mu_1,\mu_2)} \\ &\underbrace{- \max_u \langle \mu_1, u\rangle - \mathcal{R}^h(u) - \max_v \langle \mu_2, v\rangle - \mathcal{R}^v(v)}_{\mathcal{D}_3^{\text{bwd}}(\mu_1,\mu_2)}\end{aligned}$$

The dual formulation in the last line is amenable to optimization strategies from Section 5.2. We can perform a forward step with respect to $\mathcal{D}_3^{\text{fwd}}(\mu_1,\mu_2)$ and a backward step with respect to $\mathcal{D}_3^{\text{bwd}}(\mu_1,\mu_2)$. The computation of the gradient for the forward step is simple. Evaluating the proximal mapping for $\mathcal{D}_3^{\text{bwd}}(\mu_1,\mu_2)$ is also easy, since the function is separable, i.e., the proximal mappings for $\mu_1$ and $\mu_2$ can be computed independently. The computation is analogue to that of $\mathcal{D}_2^{\text{bwd}}(\mu)$.

# 6 Conclusion

In this paper, we proved a local convergence result for abstract descent methods, which is similar to that of Attouch et al. [2]. This local convergence result is applicable to an inertial forward–backward splitting method, called iPiano [29]. For functions that satisfy the Kurdyka–Łojasiewicz inequality at a local optimum, under a certain growth condition, we verified that the sequence of iterates stays in a neighborhood of a local (or global) minimum and converges to the minimum. As a consequence, the properties that imply convergence of iPiano must hold locally only. Combined with a well-known expression for the gradient of Moreau envelopes in terms of the proximal mapping, relations of iPiano to an inertial averaged proximal minimization method and an inertial alternating proximal minimization are uncovered. These considerations are conducted for functions that are prox-regular instead of the stronger assumption of convexity. For a non-convex feasibility problem, experimentally, iPiano significantly outperforms the alternating projection method and a recently proposed non-convex variant of Douglas–Rachford splitting.

# Appendix

**Lemma 13.** *Let $\mathcal{F}\colon \mathbb{R}^{2N} \to \mathbb{R}\cup\{\infty\}$ be a proper lower semi-continuous function which satisfies the Kurdyka–Łojasiewicz property at some point $z^* = (z_1^*, z_2^*) \in \mathbb{R}^{2N}$. Denote by $U$, $\eta$ and $\varphi\colon [0,\eta) \to \mathbb{R}_+$ the objects appearing in Definition 4 of the KL property at $z^*$. Let $\sigma, \rho > 0$ be such that $B(z^*,\sigma) \subset U$ with $\rho \in (0,\sigma)$, where $B(z^*,\sigma) := \{z \in \mathbb{R}^{2N} : |z-z^*| < \sigma\}$.*

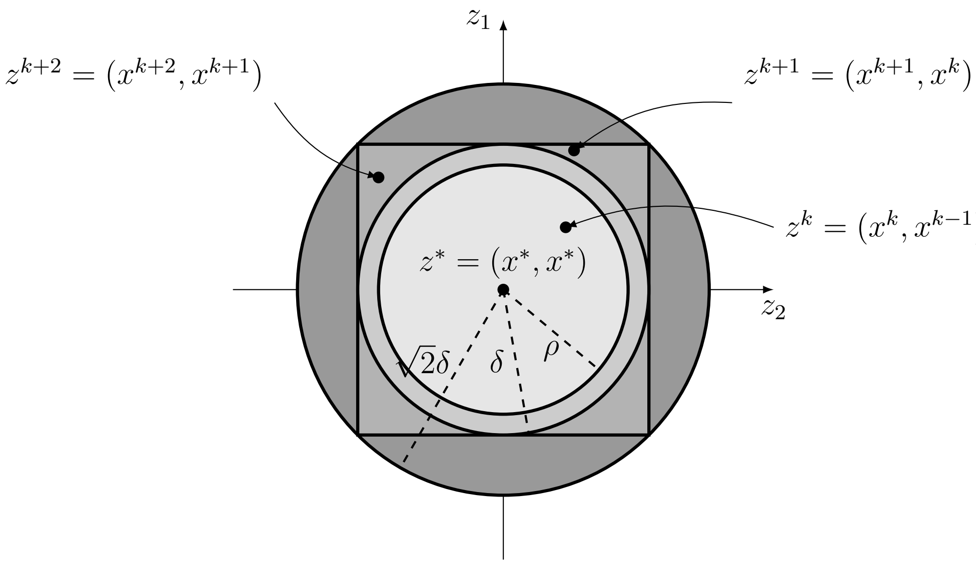


Figure 3: An essential step of the proof of Theorem 6 is to show: $z^k \in B_\rho(z^*) = B_\delta(x^*, x^*)$ implies $x^{k+2}, x^{k+1} \in B_\delta(z_2^*) = B_\delta(x^*)$ which restricts $z^{k+1}$ and $z^{k+2}$ to the rectangle in the plot and thus to $B_{\sqrt{2}\delta}(z^*)$.

*Furthermore, let $(z^k)_{k\in\mathbb{N}} = (x^k, x^{k-1})_{k\in\mathbb{N}}$ be a sequence satisfying (H1), (H2), and*

$$\forall k \in \mathbb{N}: \quad z^k \in B(z^*, \rho) \Rightarrow z^{k+1} \in B(z^*, \sigma) \text{ with } \mathcal{F}(z^{k+1}), \mathcal{F}(z^{k+2}) \geq \mathcal{F}(z^*)\,. \tag{9}$$

*Moreover, the initial point $z^0 = (x^0, x^{-1})$ is such that $\mathcal{F}(z^*) \leq \mathcal{F}(z^0) < \mathcal{F}(z^*) + \eta$ and*

$$|x^* - x^0| + \sqrt{\frac{\mathcal{F}(z^0) - \mathcal{F}(z^*)}{a}} + \frac{b}{a}\varphi(\mathcal{F}(z^0) - \mathcal{F}(z^*)) < \frac{\rho}{2}\,. \tag{10}$$

*Then, the sequence $(z^k)_{k\in\mathbb{N}}$ satisfies*

$$\forall k \in \mathbb{N}: z^k \in B(z^*, \rho), \quad \sum_{k=0}^{\infty} |x^k - x^{k-1}| < \infty, \quad \mathcal{F}(z^k) \to \mathcal{F}(z^*), \text{ as } k \to \infty\,, \tag{11}$$

*$(z^k)_{k\in\mathbb{N}}$ converges to a point $\bar{z} = (\bar{x}, \bar{x}) \in B(z^*, \sigma)$ such that $\mathcal{F}(\bar{z}) \leq \mathcal{F}(z^*)$. If, additionally, (H3) is satisfied, then $0 \in \partial\mathcal{F}(\bar{z})$ and $\mathcal{F}(\bar{z}) = \mathcal{F}(z^*)$.*

*Proof of Theorem 6* Let $r > 0$. Since $\mathcal{F}$ satisfied the KL property at $z^*$ there exist $\eta_0 \in (0, +\infty]$, $\delta \in (0, r)$ and a continuous concave function $\varphi\colon [0, \eta_0) \to \mathbb{R}$ such that $\varphi(0) = 0$, $\varphi$ is continuously differentiable and strictly increasing on $(0, \eta_0)$, and for all

$$z \in B_{\sqrt{2}\delta}(z^*) \cap [\mathcal{F}(z^*) < \mathcal{F}(z) < \mathcal{F}(z^*) + \eta_0]$$

the KL inequality holds. Due to $z^*$ being a local optimum $\delta$ can be chosen such that

$$\mathcal{F}(z) \geq \mathcal{F}(z^*) \quad \text{for all} \quad z \in B_{\sqrt{2}\delta}(z^*)\,. \tag{12}$$

We want to verify the implication in (9). Let $\eta := \min(\eta_0, \nu)$ and $k \in \mathbb{N}$. Assume $z^0, \dots, z^k \in B_\rho(z^*)$, with $z^k =: (z_1^k, z_2^k) = (x^k, x^{k-1}) \in \mathbb{R}^{N\times 2}$ and w.l.o.g. $\mathcal{F}(z^*) < \mathcal{F}(z^0), \dots, \mathcal{F}(z^k) < \mathcal{F}(z^*) + \eta$ (note that if $\mathcal{F}(z^k) = \mathcal{F}(z^*)$ the sequence is stationary and the result trivial).

See Figure 3 for the idea of the following steps. First, note that $x^k \in B_\delta(z_2^*)$ as $z^k \in B_\delta(z^*)$. Suppose $z_2^{k+2} = x^{k+1} \notin B_\delta(z_2^*)$. Then by (H4) and (H1) we observe (use $(u+v)^2 \leq 2(u^2+v^2)$)

$$\mathcal{F}(z^k) < \mathcal{F}(z^{k+2}) + \frac{a}{4}|z_2^k - z_2^{k+2}|^2 \\ \leq \mathcal{F}(z^k) - a\left(|z_2^{k+2} - z_2^{k+1}|^2 + |z_2^{k+1} - z_2^k|^2\right) + \frac{a}{4}|z_2^k - z_2^{k+2}|^2 \leq \mathcal{F}(z^k)\,,$$

which is a contradiction and therefore $z_2^{k+2} \in B_\delta(z_2^*)$. Hence, due to the equivalence of norms in finite dimensions, $z^{k+1} = (x^{k+1}, x^k) \in B_{\sqrt{2}\delta}(z^*)$. Thanks to (12), we have $\mathcal{F}(z^{k+1}) \geq \mathcal{F}(z^*)$. In order to verify (9), we also need $\mathcal{F}(z^{k+2}) \geq \mathcal{F}(z^*)$, which can be shown analogously, however we need to consider three iteration steps (that's the reason for the factor $\frac{a}{4}$ instead of $\frac{a}{2}$ on the right hand side of (H4)). Assuming $z_2^{k+3} = x^{k+2} \notin B_\delta(z_2^*)$ yields the following contradiction:

$$\begin{aligned}\mathcal{F}(z^k) &< \mathcal{F}(z^{k+3}) + \frac{a}{4}|z_2^k - z_2^{k+3}|^2 \\ &\leq \mathcal{F}(z^k) - a\left(|z_2^{k+3} - z_2^{k+2}|^2 + |z_2^{k+2} - z_2^{k+1}|^2 + |z_2^{k+1} - z_2^k|^2\right) + \frac{a}{4}|z_2^k - z_2^{k+3}|^2 \\ &\leq \mathcal{F}(z^k) - a\left(|z_2^{k+3} - z_2^{k+2}|^2 + |z_2^{k+2} - z_2^{k+1}|^2 + |z_2^{k+1} - z_2^k|^2\right) \\ &\quad + \frac{a}{4}\left(2|z_2^{k+3} - z_2^{k+2}|^2 + 4|z_2^{k+2} - z_2^{k+1}|^2 + 4|z_2^{k+1} - z_2^k|^2\right) \leq \mathcal{F}(z^k)\,.\end{aligned}$$

Therefore, $\mathcal{F}(z^{k+1}), \mathcal{F}(z^{k+2}) \geq \mathcal{F}(z^*)$ holds, which is exactly property (9) with $\sigma = \sqrt{2}\delta$.

Now, choose $u, \mu > 0$ such that

$$\mu < \eta\,,\ u < \frac{\rho}{6}\,,\ \sqrt{\frac{\mu}{a}} + \frac{b}{a}\varphi(\mu) < \frac{\rho}{3}\,.$$

If $z^0$ satisfies (3), we have

$$|z_1^* - z_1^0| + \sqrt{\frac{\mathcal{F}(z^0) - \mathcal{F}(z^*)}{a}} + \frac{b}{a}\varphi(\mathcal{F}(z^0) - \mathcal{F}(z^*)) < \frac{\rho}{2}\,,$$

which is exactly property (10). Using Lemma 13 we conclude that the sequence has the finite length property, remains in $B_\rho(z^*)$, converges to $\bar{z} \in B_\delta(z^*)$, $\mathcal{F}(z^k) \to \mathcal{F}(z^*)$ and $\mathcal{F}(\bar{z}) \leq \mathcal{F}(z^*)$, which is only allowed for $\mathcal{F}(\bar{z}) = \mathcal{F}(z^*)$. If the sequence also has property (H3) $\bar{z}$ is a critical point of $\mathcal{F}$. □

*Proof of Lemma 7* Let $\delta > \rho$ and $\nu$ be postive numbers. For $y = (y_1, y_2) \in \mathbb{R}^{2N}$ with $y_2 \notin B_\delta(z_2^*)$ and $z = (z_1, z_2) \in B_\rho(z^*)$ such that $\mathcal{F}(z) < \mathcal{F}(z^*) + \nu$, the growth condition in

Lemma 7 shows that

$$\begin{aligned}\mathcal{F}(y) &\geq \mathcal{F}(z^*) - \frac{a}{16}|y_2 - z_2^*|^2 \\ &\geq \mathcal{F}(z) - \nu - \frac{a}{8}|y_2 - z_2^*|^2 + \frac{a}{16}|y_2 - z_2^*|^2 \\ &\geq \mathcal{F}(z) - \nu - \frac{a}{4}|y_2 - z_2| - \frac{a}{4}|z_2 - z_2^*|^2 + \frac{a}{16}|y_2 - z_2^*|^2 \\ &\geq \mathcal{F}(z) - \frac{a}{4}|y_2 - z_2| + (-\nu - \frac{a}{4}\rho^2 + \frac{a}{16}\delta^2)\,,\end{aligned}$$

where for sufficiently small $\nu$ and $\rho$ the term in the parenthesis becomes positive, which implies (H4). □

*Proof of Proposition 10* Without loss of generality, we can take $\bar{x} = 0$. As $f$ is prox-bounded the condition for prox-regularity may be taken to be global, i.e., there exists $\varepsilon > 0$ and $\lambda_0 > 0$ such that

$$f(x') > f(x) + \langle v, x' - x\rangle - \frac{1}{2\lambda_0}|x' - x|^2 \quad \forall x' \neq x \tag{13}$$

$$\text{when } v \in \partial f(x),\ |v| < \varepsilon,\ |x| < \varepsilon,\ f(x) < f(0) + \varepsilon\,. \tag{14}$$

Let $T$ be the $f$-attentive localization of $\partial f$ specified in (14). For $\lambda \in (0, \lambda_0)$ and $u = x + \lambda v$ the subgradient inequality (13) implies

$$f(x') + \frac{1}{2\lambda}|x' - u|^2 > f(x) + \frac{1}{2\lambda}|x - u|^2\,.$$

Therefore, $P_\lambda f(x + \lambda v) = \{x\}$ when $v \in T(x)$. In general, for any $u$ sufficiently close to 0, thanks to Fermat's rule on the minimization problem of $P_\lambda f(u)$, we have for any $x \in P_\lambda f(u)$ that $v = (u - x)/\lambda \in T(x)$ holds. Thus, $U_\lambda = \operatorname{rge}(I + \lambda T)$ is a neighborhood of 0 on which $P_\lambda f$ is single-valued and coincides with $(I + \lambda T)^{-1}$.

Now, let $u = x + \lambda v$ and $u' = x' + \lambda v'$ be any two elements in $U_\lambda$ such that $x = P_\lambda f(u)$ and $x' = P_\lambda f(u')$. Then $(x, v)$ and $(x', v')$ belong to $\operatorname{Graph} T$. Therefore, we can add (13) and (13) with roles of $x$ and $x'$ interchanged to obtain

$$0 \geq \langle v - v', x' - x\rangle - \frac{1}{\lambda_0}|x' - x|^2\,. \tag{15}$$

In this inequality, we substitute $v$ with $(u - x)/\lambda$ and $v'$ with $(u' - x')/\lambda$ which yields

$$0 \leq \frac{1}{\lambda_0}|x' - x|^2 + \frac{1}{\lambda}\langle (u' - x') - (u - x), x' - x\rangle = \frac{1}{\lambda}\langle u' - u, x' - x\rangle + \left(\frac{1}{\lambda_0} - \frac{1}{\lambda}\right)|x' - x|^2$$

or, equivalent, using $\delta := \frac{1}{\lambda} - \frac{1}{\lambda_0} > 0$

$$\langle u' - u, x' - x\rangle \geq \lambda\delta|x' - x|^2\,.$$

This expression helps to estimate the local Lipschitz constant of the gradient of the Moreau envelope. Using the closed form description of $\nabla e_\lambda f$ on $U_\lambda$, we verify the $\lambda^{-1}$-Lipschitz continuity of $\nabla e_\lambda f$ as follows:

$$\begin{aligned}\lambda^2|\nabla e_\lambda f(u) - \nabla e_\lambda f(u')|^2 - |u-u'|^2 &= |(u-u') - (P_\lambda f(u) - P_\lambda f(u'))|^2 - |u-u'|^2 \\ &= |x-x'|^2 - 2\langle u-u', x-x'\rangle \\ &\le (1-2\lambda\delta)|x-x'|^2 \\ &\le 0\end{aligned}$$

when $\lambda\delta \ge \frac{1}{2}$, i.e., $\lambda \le \frac{1}{2}\lambda_0$. □

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