# Unifying abstract inexact convergence theorems for descent methods and block coordinate variable metric iPiano

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**Abstract**

An abstract convergence theorem for a class of descent method that explicitly models relative errors is proved. The convergence theorem generalizes and unifies several recent abstract convergence theorems, and is applicable to possibly non-smooth and non-convex lower semi-continuous functions that satisfy the Kurdyka–Łojasiewicz inequality, which comprises a huge class of problems. The descent property is measured with respect to a function that is allowed to change along the iterations, which makes block coordinate and variable metric methods amenable to the abstract convergence theorem. As a particularly algorithm, the convergence of a block coordinate variable metric version of iPiano (an inertial forward–backward splitting algorithm) is proved. The newly introduced algorithms perform favorable on an inpainting problem with a Mumford–Shah-like regularization from image processing.

***Keywords*** — *abstract convergence theorem, Kurdyka–Łojasiewicz inequality, descent method, relative errors, block coordinate method, variable metric method, inertial method, iPiano, inpainting, Mumford–Shah regularizer*

## 1 Introduction

In this work, we propose a convergence analysis for inexact abstract descent methods for the minimization of a proper lower semi-continuous (possibly non-smooth and non-convex) extended-valued function. The goal of this paper is to unify and extend the frameworks introduced by Attouch et al. [5] (and extended in [16]) and Ochs et al. [28]. Their convergence analysis is driven by two central assumptions a *sufficient decrease condition* and a *relative error condition*. While, in the former work, the idea of a decrease condition of an algorithm is to guarantee a strict decrease of the objective value until an optimal point is reached, in the latter work, this condition is applied to a Lyapunov-type function that is only known to majorize the original function. Although, the function values along the iterates of an algorithm can increase, the descent property of the majorizer enforces their convergence. Both convergence theorems are recovered as special cases of the one that is proposed in this paper.

The design of the descent property in this paper is motivated by the observation that a lot of flexibility is gained by allowing the function to change along the iterations. This can either be achieved by a "blind" change of the function as a sequence of functions, or by a more controlled version, where the function is parameter dependent. The abstract convergence results mentioned above rely on the fact that convergence of $(F(x^n))_{n\in\mathbb{N}}$ implies convergence of the sequence $(x^n)_{n\in\mathbb{N}}$. However, in the framework presented here convergence of $(F(x^n, u^n))_{n\in\mathbb{N}}$ implies convergence only for $(x^n)_{n\in\mathbb{N}}$, while—under some mild restrictions—the sequence of "parameters" $(u^n)_{n\in\mathbb{N}}$ allows for a change of the function along the iterations. Using the gained flexibility, we formulate a block coordinate variable metric version of iPiano [28], an inertial forward–backward splitting algorithm.

Like in the papers mentioned above, the convergence theory relies on a non-smooth version of the so-called Kurdyka–Łojasiewicz (KL) inequality [7, 8, 3]. It is a generalization of the Łojasiewicz inequality for real analytic functions, which was used to show convergence of bounded trajectories of a gradient dynamical system. For a real analytic function $f\colon \mathbb{R}^N \to \mathbb{R}$, it bounds $|f - f(a)|^\theta / \|\nabla f\|$ around a critical point $a \in \mathbb{R}^N$ for some $\theta \in [\frac{1}{2}, 1)$. Kurdyka extends this result to smooth functions definable in an o-minimal structure [19], and Bolte et al. [7, 8] to non-smooth functions definable in an o-minimal structure, which comprises a huge class of functions—nearly all functions appearing in practice. A "small" subclass of those functions are semi-algebraic functions (which in practice is usually big enough). In fact, o-minimal structures are an axiomatic construction that preserves the favorable properties of the semi-algebraic structure [15].

The convergence of several algorithm has been shown in recent years for possibly non-smooth and non-convex functions. Several of these results are analyzed in the general framework of KL function, and many convergence results can be put into the abstract framework of descent methods in [5]. Convergence of the gradient method is proved in [1, 5], and can be extended to proximal gradient descent (resp. forward–backward splitting method) [5], which applies to a class of problems that is given as the sum of a (possibly non-smooth and non-convex) function and a smooth (possibly non-convex) function (see also [21]). Accelerations by means of a variable metric are considered in [13, 16]. Since the proximal gradient descent method reduces to the well-known projected gradient descent method when the non-smooth function is the indicator function of a set, these results directly imply convergence of the projected gradient descent. In [26] the convergence of a subgradient-oriented method is analyzed, where the KL-inequality is observed to be in general not sufficient to guarantee convergence to a single critical point.

The convergence of proximal methods is inspected in [3, 5, 9, 24], and an alternating proximal method is considered in [4]. Extensions to block coordinate methods are given, e.g. in [5] under the name regularized Gauss–Seidel method, which is actually a variable metric version of the block coordinate methods in [4, 6, 18]. The combination of the ideas of alternating proximal minimization and forward–backward splitting can be found in [10], where the algorithm is called proximal alternating linearized minimization (PALM) (see also [31]). For an extension that allows the metric to change in each iteration with a flexible order of the block iterations we refer to [14].

Another possibility to accelerate descent methods (instead of using a variable metric) are so-called inertial methods. In convex optimization inertial or overrelaxation methods are known to be optimal [25]. Although it is hard to obtain sharp lower complexity bounds in the non-convex setting, hence to argue about optimal methods, experiment indicate that inertial algorithms are favorable. In [28, 27] an extension of inertial gradient descent (also known as Heavy-ball method or gradient descent with momentum), which includes an additional non-smooth term in the objective function alike forward–backward splitting, is analyzed in the KL framework. The proposed algorithm is called iPiano and shows good performance in applications. An abstract convergence theorem is proved that reveals similarities to the one in [5], however requires to consider three iterates at the same time instead of only the current and preceding one. In [27, 12] the original problem class "non-smooth convex plus smooth non-convex" was extended to "non-smooth non-convex plus smooth non-convex". See [11] for a slight variant of this algorithm. The algorithm iPiano relies on a hyperparameter that has the advantage of an explicit construction of feasible step sizes, which is not considered in [11, 12] where pure existence of feasible step sizes is asserted. [12] considers Bregman proximity functions in the update step. A variable metric method has not been proposed yet. The inertial term of the algorithm complicates the change of parameters or step sizes, thus in [28] the hyperparameter was required to be stationary after a finite number of iterations. Though it is not a severe restriction in practice, in this paper the issue is resolved and even a variable metric (whose convergence need not be inferred a priori) can be used. The abstract convergence theorem that we prove in this paper reveals the extension of variable metric iPiano to a block coordinate variable metric version more or less for free—it operates like PALM but with inertial term and a variable metric. The block coordinate updates are required to be essentially cyclic, i.e., the ordering of the updates is arbitrary up to the restriction that each block must be updated infinitely often.

Like in [28], our convergence analysis relies on a Lyapunov-type function that majorizes the actual objective function. This concept of proving a descent property and a relative error condition for a majorizing function was first proposed in [28]. Then, it has been used in [11, 12] for a quite similar inertial algorithm. However, this concept is important beyond inertial methods. It is used to prove convergence of Douglas–Rachford splitting [20] and Peaceman–Rachford splitting [22] for non-convex optimization problems.

The newly introduced block coordinate variable metric version of iPiano is applied to an image inpainting/compression problem, where the goal is to recover the original image from a small number of pixels. The algorithm solves a minimization problem which keeps the known pixel values unchanged and fills the missing data according to the minimization of a regularization term. Usually linear diffusion is used in this context. However, here, we consider a more general method and regularize with the Ambrosio–Tortorelli approximation of the Mumford–Shah model.

Section 2 introduces the basic notation and results from (non-smooth) variational analysis [29] and the Kurdyka–Łojasiewicz inequality. Section 3 formulates the basic assumptions for the abstract convergence theorem, which is motivated by the results in [5, 16, 28, 10]. Section 3.4 reveals the relation of our abstract convergence theorem and the ones that are generalized. The flexibility that our convergence theorem gains as compared to [5, 16, 28] is used in Section 4 to prove convergence of a variable metric version of iPiano [28] and in Section 5 of a block coordinate variable metric version of iPiano. Thanks to the formulation of the abstract convergence theorem, the block coordinate version does not require much extra work for proving its convergence. Several block coordinate, variable metric, and inertial versions of forward–backward splitting/iPiano are applied to an image inpainting problem in Section 6, which emphasizes the importance of a variable metric and block coordinate methods.

# 2 Preliminaries

## 2.1 Notation and definitions

Throughout this paper, we will always work in a finite dimensional Euclidean vector space $\mathbb{R}^N$ of dimension $N \in \mathbb{N}$, where $\mathbb{N} := \{1, 2, \dots\}$. The vector space is equipped with the standard Euclidean norm $\|\cdot\| := \|\cdot\|_2$ that is induced by the standard Euclidean inner product $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. If specified explicitly, we work in a metric induced by a symmetric positive definite matrix $A \in \mathbb{S}_{++}(N) \subset \mathbb{R}^{N\times N}$, represented by the inner product $\langle x, y\rangle_A := \langle Ax, y\rangle$ and the norm $\|x\|_A := \sqrt{\langle x, x\rangle_A}$. For $A \in \mathbb{S}_{++}(N)$ we define $\varsigma(A) \in \mathbb{R}$ as the largest value that satisfies $\|x\|_A^2 \geq \varsigma(A)\|x\|_2^2$ for all $x \in \mathbb{R}^N$.

As usual, we consider extended read-valued functions $f : \mathbb{R}^N \to \overline{\mathbb{R}}$, $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$, that are defined on the whole space with *domain* given by $\operatorname{dom} f := \{x \in \mathbb{R}^N \,|\, f(x) < +\infty\}$. A function is called *proper* if it is nowhere $-\infty$ and not everywhere $+\infty$. We define the *epigraph* of the function $f$ as $\operatorname{epi} f := \{(x, \mu) \in \mathbb{R}^{N+1} \,|\, \mu \geq f(x)\}$. We will also need to consider set-valued mappings $F : \mathbb{R}^N \rightrightarrows \mathbb{R}^M$ defined by the *graph*

$$\operatorname{Graph} F := \{(x, y) \in \mathbb{R}^N \times \mathbb{R}^M \,|\, y \in F(x)\} ,$$

where the domain of a set-valued mapping is given by $\operatorname{dom} F := \{x \in \mathbb{R}^N \,|\, F(x) \neq \emptyset\}$. For a proper function $f : \mathbb{R}^N \to \overline{\mathbb{R}}$ we define the set of *(global) minimizers* as

$$\arg\min f := \arg\min_{x\in\mathbb{R}^N} f := \{x \in \mathbb{R}^N \,|\, f(x) = \inf f\} , \qquad \inf f := \inf_{x\in\mathbb{R}^N} f(x) .$$

The *Fréchet subdifferential* of $f$ at $\bar{x} \in \operatorname{dom} f$ is the set $\widehat{\partial} f(\bar{x})$ of those elements $v \in \mathbb{R}^N$ such that

$$\liminf_{\substack{x \to \bar{x} \\ x \neq \bar{x}}} \frac{f(x) - f(\bar{x}) - \langle v, x - \bar{x} \rangle}{\|x - \bar{x}\|} \geq 0$$

For $\bar{x} \notin \operatorname{dom} f$, we set $\widehat{\partial} f(\bar{x}) = \emptyset$. For convenience, we introduce $f$-*attentive convergence*: A sequence $(x^n)_{n \in \mathbb{N}}$ is said to $f$-*converge* to $\bar{x}$ if

$$x^n \to \bar{x} \quad \text{and} \quad f(x^n) \to f(\bar{x}) \quad \text{as } n \to \infty\,,$$

and we write $x^n \xrightarrow{f} \bar{x}$. The so-called *(limiting) subdifferential* of $f$ at $\bar{x} \in \operatorname{dom} f$ is defined by

$$\partial f(\bar{x}) := \{ v \in \mathbb{R}^N \,|\, \exists\, x^n \xrightarrow{f} \bar{x},\ v^n \in \widehat{\partial} f(x^n),\ v^n \to v \}\,,$$

and $\partial f(\bar{x}) = \emptyset$ for $\bar{x} \notin \operatorname{dom} f$. A point $\bar{x} \in \operatorname{dom} f$ for which $0 \in \partial f(\bar{x})$ is a called a *critical points*. As a direct consequence of the definition of the limiting subdifferential, we have to following closedness property:

$$x^n \xrightarrow{f} \bar{x},\ v^n \to \bar{v},\ \text{and for all } n \in \mathbb{N}\colon v^n \in \partial f(x^n) \quad \Longrightarrow \quad \bar{v} \in \partial f(\bar{x})\,.$$

[29, Ex. 8.8] shows that at a point $\bar{x} \in \mathbb{R}^N$, for the sum of an extended-valued function $g$ that is finite at $\bar{x}$ and a function that is continuously differentiable (smooth) function $f$ around $\bar{x}$ it holds that $\partial(g + f)(\bar{x}) = \partial g(\bar{x}) + \nabla f(\bar{x})$. Moreover for a function $f\colon \mathbb{R}^N \times \mathbb{R}^M \to \overline{\mathbb{R}}$ with $f(x, y) = f_1(x) + f_2(y)$ the subdifferential satisfies $\partial f(x, y) = \partial f_1(x) \times \partial f_2(y)$ [29, Prop. 10.5].

Finally, the *distance* of $\bar{x} \in \mathbb{R}^N$ to a set $\omega \subset \mathbb{R}^N$ as is given by $\operatorname{dist}(\bar{x}, \omega) := \inf_{x \in \omega} \|\bar{x} - x\|$ and we introduce $\|\partial f(\bar{x})\|_- := \inf_{v \in \partial f(\bar{x})} \|v\| = \operatorname{dist}(0, \partial f(\bar{x}))$ what is known as the *lazy slope* of $f$ at $\bar{x}$. Note that $\inf \emptyset := +\infty$ by definition. Furthermore, we have (see [16]):

**Lemma 1.** *If $x^n \xrightarrow{f} \bar{x}$ and $\liminf_{n \to \infty} \|\partial f(x^n)\|_- = 0$, then $0 \in \partial f(\bar{x})$.*

For a function $f$, we use the notation (analogously for other constraints)

$$[f < \mu] := \{ x \in \mathbb{R}^N \,|\, f(x) < \mu \}\,.$$

## 2.2 The Kurdyka–Łojasiewicz property

**Definition 2** (Kurdyka–Łojasiewicz property / KL property)**.** *Let $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ be an extended real valued function and let $\bar{x} \in \operatorname{dom} \partial f$. If there exists $\eta \in (0, \infty]$, a neighborhood $U$ of $\bar{x}$ and a continuous concave function $\varphi\colon [0, \eta) \to \mathbb{R}_+$ such that*

$$\varphi(0) = 0, \quad \varphi \in C^1((0, \eta)), \quad \textit{and} \quad \varphi'(s) > 0 \textit{ for all } s \in (0, \eta),$$

*and for all $x \in U \cap [f(\bar{x}) < f(x) < f(\bar{x}) + \eta]$ holds the Kurdyka–Łojasiewicz inequality*

$$\varphi'(f(x) - f(\bar{x})) \|\partial f(x)\|_- \geq 1\,, \tag{1}$$

*then the function has the Kurdyka–Łojasiewicz property at $\bar{x}$.*

*If, additionally, the function is lower semi-continuous and the property holds for each point in $\operatorname{dom} \partial f$, then $f$ is called a Kurdyka–Łojasiewicz function.*

Figure 1, which is taken from [27], shows the idea and the variables appearing in the definition of the KL property for a smooth function. For smooth functions (assume $f(\bar{x}) = 0$), (1) reduces to $\|\varphi \circ f\| \geq 1$ around the point $\bar{x}$, which means that after reparametrization with a *desingularization function* $\varphi$ the function is sharp. "Since the function $\varphi$ is used here to turn a singular region—a region in which the gradients are arbitrarily small—into a regular region, i.e. a place where the gradients are bounded away from zero, it is called a desingularization function for $f$." [5]. It is easy to see that the KL property is satisfied for all non-stationary points [4].

The KL property is satisfied by a large class of functions, namely functions that are definable in an o-minimal structure (see [4, Thm. 14] and [8, Thm. 14]).

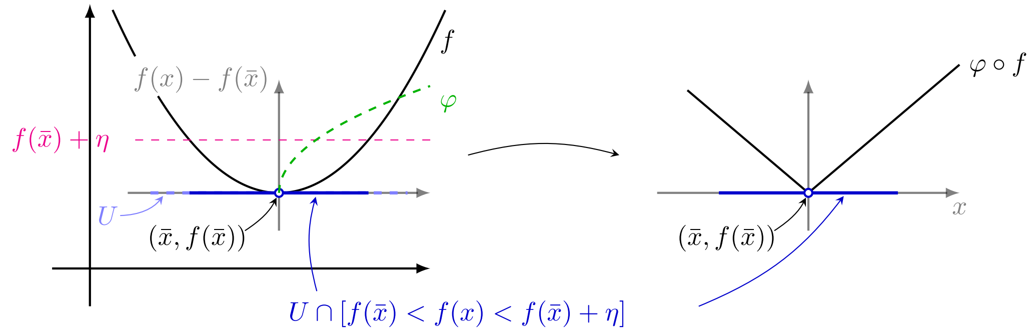


Figure 1: Example of the KL property for a smooth function. The composition $\varphi \circ f$ has a slope of magnitude 1 except at $\hat{x}$.

**Theorem 3** (Nonsmooth Kurdyka–Łojasiewicz inequality for definable functions)**.** *Any proper lower semi-continuous function $f\colon X \to \overline{\mathbb{R}}$ which is definable in an* o*-minimal structure $\mathcal{O}$ has the Kurdyka–Łojasiewicz property at each point of* $\operatorname{dom}\partial f$*. Moreover the function $\varphi$ in Definition 2 is definable in $\mathcal{O}$.*

In particular, semi-algebraic and globally subanalytic sets and functions are definable in such a structure. There is even an o-minimal structure that extends the one of globally subanalytic functions with the exponential function (thus also the logarithm is included) [30, 15]. As mentioned in the introduction, o-minimal structures can be seen as an axiomatization of the nice properties of semi-algebraic functions, and are therefore designed such that the structure is preserved under many operations, for example, pointwise addition and multiplication, composition and inversion. A brief summary of the concepts that are important for this paper can be found in [4] (or [27, Section 4.5]).

Before we introduce the general framework and the convergence analysis in the next sections, let us first consider a so-called *uniformization results*, which was proved in [3] for the Łojasiewicz property and adjusted in [10] for the KL property. Its main implication for this paper—like in [10]—is that it allows for a direct proof of the main convergence theorem without the need of an induction argument.

**Lemma 4** (Uniformization result [10])**.** *Let $\omega$ be a compact set and let $f\colon \mathbb{R}^d \to \overline{\mathbb{R}}$ be a proper and lower semi-continuous function. Assume that $f$ is constant on $\omega$ and satisfies the KL property at each point of $\omega$. Then, there exist $\varepsilon > 0$, $\eta > 0$, and a continuous concave function $\varphi\colon [0, \eta) \to \mathbb{R}_+$ such that*

$$\varphi(0) = 0, \quad \varphi \in C^1((0,\eta)), \quad \textit{and} \quad \varphi'(s) > 0 \textit{ for all } s \in (0,\eta),$$

*such that for all $\bar{x} \in \omega$ and all $x$ in the following intersection*

$$[\operatorname{dist}(x,\omega) < \varepsilon] \cap [f(\bar{x}) < f(x) < f(\bar{x}) + \eta] \tag{2}$$

*one has,*

$$\varphi'(f(x) - f(\bar{x}))\|\partial f(x)\|_- \geq 1\,.$$

# 3 An abstract inexact convergence theorem

In this section, let $F\colon \mathbb{R}^N \times \mathbb{R}^P \to \overline{\mathbb{R}}$ be a proper, lower semi-continuous function that is bounded from below. We analyze convergence of an abstract algorithm that generates a sequence $(x^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ under the following assumptions:

**Assumption 5.** *Let $(u^n)_{n\in\mathbb{N}}$ be a sequence of parameters in $\mathbb{R}^P$, and let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an $\ell_1$-summable sequence of non-negative real numbers. Moreover, we assume there are sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$, and $(d_n)_{n\in\mathbb{N}}$ of non-negative real numbers such that the following holds:*

*(H1) (Sufficient decrease condition) For each $n \in \mathbb{N}$, it holds that*

$$F(x^{n+1}, u^{n+1}) + a_n d_n^2 \le F(x^n, u^n)\,.$$

*(H2) (Relative error condition) For each $n \in \mathbb{N}$, the following holds:*

$$b_{n+1} \|\partial F(x^{n+1}, u^{n+1})\|_- \le \frac{b}{2}(d_{n+1} + d_n) + \varepsilon_{n+1}\,.$$

*(H3) (Continuity condition) There exists a subsequence $((x^{n_j}, u^{n_j}))_{j\in\mathbb{N}}$ and $(\tilde{x}, \tilde{u}) \in \mathbb{R}^N$ such that*

$$(x^{n_j}, u^{n_j}) \xrightarrow{F} (\tilde{x}, \tilde{u}) \quad \textit{as} \quad j \to \infty\,.$$

*(H4) (Contraction condition) It holds that*

$$\|x^{n+1} - x^n\|_2 \in \mathrm{o}(d_n) \qquad \textit{and} \quad (b_n)_{n\in\mathbb{N}} \notin \ell_1\,, \quad \sup_{n\in\mathbb{N}} b_n a_n < \infty\,, \quad \inf_n a_n =: \underline{a} > 0\,.$$

*Remark* 1. (H4) implies that if $a_n = 0$ for all $n \ge n_0$ for some $n_0 \in \mathbb{N}$, then (due to $\|x^{n+1} - x^n\|_2 \le c a_n$ for some $c \in \mathbb{R}$) we have $x^{n+1} = x^n$ for all $n \ge n_0$.

*Remark* 2.

- The parametrization of $a_n$ and $b_n$ in the conditions (H1) and (H2) is inspired by [16].
- Note that $(a_n)_{n\in\mathbb{N}}$ is not a priori assumed to be bounded. We obtain feasible sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ when $a_n \to \infty$ and $b_n \to 0$ (not too fast due to $(b_n)_{n\in\mathbb{N}} \notin \ell_1$) such that $\sup_{n\in\mathbb{N}} b_n a_n < \infty$.
- Let us intuitively consider the interaction between the conditions. The situation $a_n \to \infty$ results in either an (asymptotically) fast decrease of $(F(x^n, u^n))_{n\in\mathbb{N}}$ or of $(\|x^{n+1} - x^n\|_2)_{n\in\mathbb{N}}$ (by the decrease of $(d_n)_{n\in\mathbb{N}}$) due to (H1). A fast decrease of $(d_n)_{n\in\mathbb{N}}$ implies a fast decrease of $(b_{n+1}\|\partial F(x^{n+1}, u^{n+1})\|_-)_{n\in\mathbb{N}}$ due to (H2). Nevertheless as $b_n \to 0$, the relative error given by $(\|\partial F(x^{n+1}, u^{n+1})\|_-)_{n\in\mathbb{N}}$ can grow (slowly) towards infinity.

## 3.1 Direct consequences of the descent property

Sufficient decrease (H1) of a certain quantity that can be related to the objective function value is key for the convergence analysis. The following lemma lists a few simple but favorable properties for such sequences.

**Lemma 6.** *Let Assumption 5 hold. Then*

*(i)* $(F(x^n, u^n))_{n\in\mathbb{N}}$ *is monotonically non-increasing,*

*(ii)* $(F(x^n, u^n))_{n\in\mathbb{N}}$ *converges,*

*(iii)* $\sum_{k=1}^n d_k^2 < +\infty$ *and, therefore,* $d_n \to 0$ *and* $\|x^{n+1} - x^n\|_2 \to 0$*, as* $n \to \infty$*.*

*Proof.* (i) and (ii) follow from (H1) and the boundedness from below of $F$. (iii) is shown by summing (H1) from $k = 1, \dots, n$ and then applying (H4):

$$\underline{a} \sum_{k=1}^n d_k^2 \le \sum_{k=1}^n a_k d_k^2 \le \sum_{k=1}^n F(x^k, u^k) - F(x^{k+1}, u^{k+1}) = (F(x^1, u^1) - \inf_{(x,u)\in\mathbb{R}^N\times\mathbb{R}^P} F(x,u)) < +\infty\,. \qquad \square$$

## 3.2 Direct consequences for the set of limit points

Like in [10], we can verify some results about the set of limit points (that depends on a certain initialization) of a bounded sequence $((x^n, u^n))_{n\in\mathbb{N}}$

$$\omega(x^0, u^0) := \limsup_{n\to\infty} \{(x^n, u^n)\}\,.$$

This definition uses the outer set-limit of a sequence of singletons, which is the same as the set of cluster points in a different notation. Moreover, we denote by $\overline{\omega}(x^0, u^0)$ the subset of limit points that allow for subsequences along which $F$ is continuous, i.e.,

$$\overline{\omega}(x^0, u^0) := \{(\bar{x}, \bar{u}) \in \omega(x^0, u^0) |\, (x^{n_j}, u^{n_j}) \overset{F}{\to} (\bar{x}, \bar{u}) \text{ for } j \to \infty\}\,.$$

We collect a few results that are of independent interest, but are required for proving our convergence result.

**Lemma 7.** *Let Assumption 5 hold and let $((x^n, u^n))_{n\in\mathbb{N}}$ be a bounded sequence.*

*(i) The set $\overline{\omega}(x^0, u^0)$ is non-empty and the set $\omega(x^0, u^0)$ is non-empty and compact.*

*(ii) $F$ is constant and finite on $\overline{\omega}(x^0, u^0)$.*

*Proof.* (i) By (H3), there exist a subsequence $((x^{n_j}, u^{n_j}))_{j\in\mathbb{N}}$ of $((x^n, u^n))_{n\in\mathbb{N}}$ that converges to $(\tilde{x}, \tilde{u})$, where at the same time the function values along this subsequence converge to $F(\tilde{x}, \tilde{u})$, therefore $\lim_{j\to\infty}(x^{n_j}, u^{n_j}) \in \overline{\omega}(x^0, u^0)$, thus $\overline{\omega}(x^0, u^0)$ is non-empty. The non-emptyness of $\omega(x^0, u^0)$ is clear and the compactness of $\omega(x^0, u^0)$ is direct consequence of its definition as an outer set-limit and the boundedness of $((x^n, u^n))_{n\in\mathbb{N}}$.

(ii) By Lemma 6(ii) $(F(x^n, u^n))_{n\in\mathbb{N}}$ converges to some $\tilde{F} \in \mathbb{R}$. For any $(\bar{x}, \bar{u}) \in \overline{\omega}(x^0, u^0)$ there exists a subsequence $((x^{n_j}, u^{n_j}))_{j\in\mathbb{N}}$ that $F$-converges to $(\bar{x}, \bar{u})$, therefore,

$$\tilde{F} = \lim_{j\to\infty} F(x^{n_j}, u^{n_j}) = F(\bar{x}, \bar{u})\,,$$

which shows that $F$ is constant on $\overline{\omega}(x^0, u^0)$.

□

**Lemma 8.** *Let Assumption 5 hold, let $((x^n, u^n))_{n\in\mathbb{N}}$ be a bounded sequence and denote by $\Pi_x(\omega) = \{x \in \mathbb{R}^N |\, (x, u) \in \omega\}$ the projection of $\omega \in \mathbb{R}^N \times \mathbb{R}^P$ onto the first $N$ coordinate dimensions. Then, we have the following results:*

*(i) The set $\Pi_x\left(\omega(x^0, u^0)\right)$ is connected.*

*(ii) If $(u^n)_{n\in\mathbb{N}}$ converges, then the set $\omega(x^0, u^0)$ is connected.*

*(iii) It holds that*

$$\lim_{n\to\infty} \operatorname{dist}((x^n, u^n), \omega(x^0, u^0)) = 0\,.$$

*Proof.* (i) is a simple application of the connectedness results [10, Lemma 3.5] and the fact that $\|x^{n+1} - x^n\|_2 \to 0$ for $n \to \infty$. (ii) follows in almost the same manner, as convergence of $u^n$ implies $\|u^{n+1} - u^n\|_2 \to 0$ as $n \to \infty$. (iii) is a direct consequence of the definition of the set of limit points. □

**Lemma 9.** *Let Assumption 5 hold, let $((x^n, u^n))_{n\in\mathbb{N}}$ be a bounded sequence and let $\sum_{n=1}^{\infty} d_n < \infty$. Then, the set $\overline{\omega}(x^0, u^0) \subset \operatorname{crit} F$.*

*Proof.* Let $(\bar{x},\bar{u}) \in \omega(x^0,u^0)$. Then, since $(b_n)_{n\in\mathbb{N}} \notin \ell_1$ from (H2), $(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_1$ and

$$\sum_{n=0}^{\infty} b_n \|\partial F(x^n,u^n)\|_- \leq \frac{b}{2}\sum_{n=0}^{\infty}(d_n + d_{n-1}) + \sum_{n=0}^{\infty}\varepsilon_n < \infty$$

follows $\liminf_{n\to\infty} \|\partial F(x^n,u^n)\|_- = 0$ and Lemma 1 yields $0 \in \partial F(\bar{x},\bar{u})$. For $(\bar{x},\bar{u}) \in \overline{\omega}(x^0,u^0)$ the subsequence $((x^{n_j},u^{n_j}))_{j\in\mathbb{N}}$ $F$-converges to $(\bar{x},\bar{u})$ as $j\to\infty$ and thus the statement follows. □

**Corollary 10.** *Let Assumption 5 hold and let $((x^n,u^n))_{n\in\mathbb{N}}$ be a bounded sequence. Suppose $F$ is continuous on the set $C\cap \operatorname{dom} F$ with an open set $C \supset \omega(x^0,u^0)$ (e.g. $F$ is continuous on $\operatorname{dom} F$), then*

$$\omega(x^0,u^0) = \overline{\omega}(x^0,u^0)\,.$$

*Proof.* Let $(x^{n_j},u^{n_j}) \to (\bar{x},\bar{u}) \in \omega(x^0,u^0)$ as $j\to\infty$. There is a neighborhood $V\subset C$ with $(\bar{x},\bar{u})\in V$ such that $(x^{n_j},u^{n_j}) \in V\cap \operatorname{dom} F$ for sufficiently large $j\in\mathbb{N}$ and continuity of $F$ implies $(x^{n_j},u^{n_j}) \xrightarrow{F} (\bar{x},\bar{u})$, thus $\omega(x^0,u^0)\subset\overline{\omega}(x^0,u^0)$. The converse inclusion holds by definition. □

## 3.3 The convergence theorem

**Theorem 11.** *Suppose $F$ is a proper lower semi-continuous Kurdyka–Lojasiewicz function that is bounded from below. Let $(x^n)_{n\in\mathbb{N}}$ be a bounded sequence generated by an abstract algorithm parametrized by a sequence $(u^n)_{n\in\mathbb{N}}$ that satisfies Assumption 5. Let $\omega(x^0,u^0) = \overline{\omega}(x^0,u^0)$.*
*Then, the following holds:*

*(i) The sequence $(x^n)_{n\in\mathbb{N}}$ satisfies*

$$\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|_2 < +\infty\,, \tag{3}$$

*and $(x^n)_{n\in\mathbb{N}}$ converges to $\tilde{x}$.*

*(ii) Moreover, if $(u^n)_{n\in\mathbb{N}}$ is a converging sequence, then $((x^n,u^n))_{n\in\mathbb{N}}$ $F$-converges to $(\tilde{x},\tilde{u})$, and $(\tilde{x},\tilde{u})$ is a critical point of $F$.*

*Proof.* By (H3) there exists a subsequence $((x^{n_j},u^{n_j}))_{j\in\mathbb{N}}$ such that $(x^{n_j},u^{n_j}) \xrightarrow{F} (\tilde{x},\tilde{u})$ as $j\to\infty$. If there is $n'$ such that $F(x^{n'},u^{n'}) = F(\tilde{x},\tilde{u})$, then (H1) implies that $F(x^n,u^n) = F(\tilde{x},\tilde{u})$, thus also $a_n = 0$, for all $n\geq n'$. Therefore, (H4) shows that $x^{n+1} = x^n$ for all $n \geq n'$, and by induction $(x^n)_{n\in\mathbb{N}}$ gets stationary and the statement is obvious (cf. Remark 1).

Now, we can assume that $F(x^n,u^n) > F(\tilde{x},\tilde{u})$ for all $n\in\mathbb{N}$. Moreover, non-increasingness of $(F(x^n,u^n))_{n\in\mathbb{N}}$ by (H1) implies that for all $\eta > 0$ there exists $n_1 \in \mathbb{N}$ such that $F(\tilde{x},\tilde{u}) < F(x^n,u^n) < F(\tilde{x},\tilde{n}) + \eta$ for all $n\geq n_1$. By definition there is also a region of attraction for the sequence $(x^n,u^n)_{n\in\mathbb{N}}$, i.e., for all $\varepsilon > 0$ there exists $n_2\in\mathbb{N}$ such that $\operatorname{dist}((x^n,u^n),\omega(x^0,u^0)) < \varepsilon$ holds for all $n\geq n_2$. In total, we know that for all $n \geq n_0 := \max\{n_1,n_2\}$ the sequence $((x^n,u^n))_{n\in\mathbb{N}}$ lies in the set

$$[F(\tilde{x},\tilde{u}) < F(x,u) < F(\tilde{x},\tilde{u}) + \eta] \cap [\operatorname{dist}((x,u),\omega(x^0,u^0)) < \varepsilon]\,.$$

Combining the facts that $\omega(x^0,u^0) = \overline{\omega}(x^0,u^0)$ is nonempty and compact from Lemma 7(i) with $F$ being finite and constant on $\omega(x^0,u^0)$ from Lemma 7(ii), allows us to apply Lemma 4 with $\omega = \omega(x^0,u^0)$. Therefore, there are $\varphi$, $\eta$, $\varepsilon$ as in Lemma 4 such that for $n > n_0$

$$\varphi'(F(x^n,u^n) - F(\tilde{x},\tilde{u}))\|\partial F(x^n,u^n)\|_- \geq 1 \tag{4}$$

holds on $\omega$. Plugging (H2) into (4) yields

$$\varphi'(F(x^n,u^n) - F(\tilde{x},\tilde{u})) \geq b_n\left(\frac{b}{2}(d_n + d_{n-1}) + \varepsilon_n\right)^{-1}\,. \tag{5}$$

By concavity of $\varphi$, i.e., (let $m > n$)

$$D^{\varphi}_{n,m} := \varphi(F(x^n, u^n) - F(\tilde{x}, \tilde{u})) - \varphi(F(x^m, u^m) - F(\tilde{x}, \tilde{u})) \geq \varphi'(F(x^n, u^n) - F(\tilde{x}, \tilde{u}))(F(x^n, u^n) - F(x^m, u^m)),$$

using (5) and (H1), we infer

$$D^{\varphi}_{n,n+1} \geq \frac{b_n a_n d_n^2}{\frac{b}{2}(d_n + d_{n-1}) + \varepsilon_n} \quad \Leftrightarrow \quad d_n^2 \leq \left(\frac{1}{2}(d_n + d_{n-1}) + \frac{\varepsilon_n}{b}\right)\left(\frac{b}{a_n b_n} D^{\varphi}_{n,n+1}\right).$$

Applying $2\sqrt{\alpha\beta} \leq \alpha + \beta$ for all $\alpha, \beta \geq 0$, we obtain (set $\varepsilon'_n := \varepsilon_n / b$ and $c := \sup_n \frac{b}{a_n b_n} < \infty$ (by (H4)))

$$2d_n \leq \frac{b}{a_n b_n} D^{\varphi}_{n,n+1} + \frac{1}{2}(d_n + d_{n-1}) + \varepsilon'_n \leq c D^{\varphi}_{n,n+1} + \frac{1}{2}(d_n + d_{n-1}) + \varepsilon'_n .$$

Now summing this inequality from $k = n_0 + 1, \ldots, n$ shows

$$\begin{aligned} 2\sum_{k=n_0+1}^{n} d_k &\leq \frac{1}{2}\sum_{k=n_0+1}^{n}\Big(d_k + d_{k-1}\Big) + c\sum_{k=n_0+1}^{n} D^{\varphi}_{k,k+1} + \sum_{k=n_0+1}^{n} \varepsilon'_k \\ &\leq \frac{1}{2} d_{n_0} + \sum_{k=n_0+1}^{n} d_k + c D^{\varphi}_{n_0+1,n+1} + \sum_{k=n_0+1}^{n} \varepsilon'_k . \end{aligned}$$

Rearranging terms and bounding $D^{\varphi}_{n_0+1,n+1} \leq \varphi(F^{n_0+1}(x^{n_0+1}) - F(\tilde{x}))$ due to $\varphi \geq 0$ results in

$$\sum_{k=n_0+1}^{n} d_k \leq \frac{1}{2} d_{n_0} + c\varphi(F^{n_0+1}(x^{n_0+1}) - F(\tilde{x})) + \sum_{k=n_0+1}^{n} \varepsilon'_k .$$

By assumption $(\varepsilon_n)_{n\in\mathbb{N}} \in \ell_1$ and therefore the right hand side is finite for any $n \geq n_0 + 1$, i.e.

$$\lim_{n\to\infty} \sum_{k=0}^{n} d_k < +\infty .$$

Thanks to (H4), for any fixed $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\|x^{n+1} - x^n\|_2 \leq \varepsilon d_n$ for all $n \geq n_0$. Combining this property with the preceding bound for the series over $d_n$, we infer for any $n \geq n_0$

$$\sum_{k=0}^{n_0} \|x^{n+1} - x^n\|_2 + \sum_{k=n_0+1}^{n} \|x^{n+1} - x^n\|_2 \leq \sum_{k=0}^{n_0} \|x^{n+1} - x^n\|_2 + \varepsilon \sum_{k=n_0+1}^{n} d_n < +\infty .$$

This property obviously shows that $(x^n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and therefore $x^n \to \tilde{x}$ as $n \to \infty$, which verifies Item (i). Item (ii) is a direct consequence of Lemma 9. □

## 3.4 Relation to other abstract convergence theorems

The abstract inexact convergence theorem in this paper generalizes [28] at least as much as [16] generalizes [5]. Moreover, our generalization comprises the results of [16].

**Relation to [5].** Our abstract convergence theorem generalizes the one from Attouch et al. [5], i.e., if a sequence $(x^n)_{n\in\mathbb{N}}$ satsifies conditions (H1)–(H3) from [5] for a proper lsc function $f\colon \mathbb{R}^N \to \overline{\mathbb{R}}$, then our Assumption 5 is also valid. Set $F(x^n, u^n) = f(x^n)$, $u^n = 0$, $a_n = a \in \mathbb{R}$, $b_n = 1$, $\varepsilon_n = 0$ for all $n \in \mathbb{N}$, and $d_n = \|x^{n+1} - x^n\|_2$, then

$$F(x^{n+1}, u^{n+1}) + a_n d_n^2 = f(x^{n+1}) + a\|x^{n+1} - x^n\|_2^2 \leq f(x^n) = F(x^n)$$

and the existence of $w^{n+1} \in \partial f(x^{n+1})$ for all $n \in \mathbb{N}$ implies (replace $b$ of our (H2) by $b' := 2b$)

$$\|\partial F(x^{n+1}, u^{n+1})\|_{-} \leq \|w^{n+1}\|_2 \leq b\|x^{n+1}-x^n\|_2 \leq b(\|x^{n+1}-x^n\|_2+\|x^{n+2}-x^{n+1}\|_2) = \frac{b'}{2}(d_n+d_{n+1})+\varepsilon_{n+1} \, .$$

Finally, the continuity assumption transfers easily (by definition of $F$)

$$x^{n_j} \to \tilde{x} \quad \text{and} \quad f(x^{n_j}) \to f(\tilde{x}) \quad \Rightarrow \quad x^{n_j} \to \tilde{x} \quad \text{and} \quad F(x^{n_j}, u^{n_j}) \to F(\tilde{x}, \tilde{u}) \, .$$

**Relation to [16].** This relation follows immediately from the relation to [5] and the design of our parameters like in [16]. Our relative error condition (H2) is more general and we allow for a second argument of the objective function $u^n$ whose convergence is not sought in the end, thus we allow for a controlled change of the objective function along the iterations.

**Relation to [28].** The abstract convergence theorem of [28] applies to a sequence $(z^n)_{n\in\mathbb{N}}$ given by $z^n = (x^n, x^{n-1})$ for a function $f\colon \mathbb{R}^{2N} \to \overline{\mathbb{R}}$. It is recovered from our framework by setting $F(z^n, u^n) = f(z^n)$, $d_n = \|x^n - x^{n-1}\|_2$, $a_n = a \in \mathbb{R}$, $b_n = 1$, and $\varepsilon_n = 0$ for all $n \in \mathbb{N}$.

# 4 Variable metric iPiano

We consider a structured nonsmooth, nonconvex optimization problem with a proper lower semi-continuous extended valued function $h\colon \mathbb{R}^N \to \overline{\mathbb{R}}$, $N \geq 1$:

$$\min_{x\in\mathbb{R}^N} \; h(x) = \min_{x\in\mathbb{R}^N} \; f(x) + g(x) \, . \tag{6}$$

The function $f\colon \mathbb{R}^N \to \mathbb{R}$ is assumed to be $C^1$-smooth (possibly nonconvex) with $L$-Lipschitz continuous gradient on $\operatorname{dom} g$, $L > 0$. Further, let the function $g\colon \mathbb{R}^N \to \overline{\mathbb{R}}$ be simple (possibly nonsmooth and nonconvex) and prox-bounded, i.e., there exists $\lambda > 0$ such that

$$e_\lambda g(x) := \inf_{y\in\mathbb{R}^N} g(y) + \frac{1}{2\lambda}\|y - x\|^2 > -\infty$$

for some $x \in \mathbb{R}^N$. Simple refers to the fact that the associated proximal map can be solved efficiently for the global optimum. Furthermore, we require $h$ to be coercive and bounded from below by some value $\underline{h} > -\infty$.

We want to obtain improved step size rules when $g$ is, for example, convex. Therefore, if $g$ is semi-convex with respect to the metric induced by $A \in \mathbb{S}_{++}(N)$, let $m$ be the semi-convexity parameter, i.e., $m \in \mathbb{R}$ is the largest value such that $g(x) - \frac{m}{2}\|x\|_A^2$ is convex. For convex functions $m = 0$ and for strongly convex functions $m > 0$. Our analysis allows us to treat also non-convex functions $g$ that are not semi-convex, by means of a "flag variable" $\sigma \in \{0, 1\}$, which is 1 if $g$ is semi-convex and 0 otherwise. Note that if $\sigma = 1$ the property of semi-convexity is satisfied for any $A \in \mathbb{S}_{++}(N)$, but with possibly changing modulus. Therefore, sometimes the metric is not explicitly specified.

The following Algorithm 1 seeks for a critical point $x^* \in \operatorname{dom} h$ of $h$, which in this case is characterized by

$$-\nabla f(x^*) \in \partial g(x^*) \, ,$$

where $\partial g$ denotes the limiting subdifferential.

**Lemma 12.** *A necessary condition for the sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ to satisfy $\gamma_n \geq c > 0$ for all $n \in \mathbb{N}$ is*

$$\alpha_n \leq \frac{1 + \sigma - 2\beta_n}{L_n - \sigma m_n + c} \quad \text{and} \quad \beta_n \leq \frac{1 + \sigma}{2} \, .$$

**Algorithm 1.** ***V****ariable* ***m****etric* ***i****nertial* ***p****ro****x****imal* ***a****lgorithm for* ***n****onconvex* ***o****ptimization (vmiPiano)*

- ***Parameter****: Let*
  - $(\alpha_n)_{n\in\mathbb{N}}$ *be a sequence of positive step size parameters,*
  - $(\beta_n)_{n\in\mathbb{N}}$ *be a sequence of non-negative parameters, and*
  - $(A_n)_{n\in\mathbb{N}}$ *be a sequence of matrices* $A_n \in \mathbb{S}_{++}(N)$ *such that* $A_n \preceq \mathrm{id}$ *and* $\inf_n \varsigma(A_n) > 0$.
  - *Let* $\sigma = 1$ *if* $g$ *is semi-convex and* $\sigma = 0$ *otherwise.*
- ***Initialization****: Choose a starting point* $x^0 \in \mathrm{dom}\, h$ *and set* $x^{-1} = x^0$.
- ***Iterations*** $(n \geq 0)$*: Update:*

$$\begin{aligned} y^n &= x^n + \beta_n(x^n - x^{n-1}) \\ x^{n+1} &\in \arg\min_{x\in\mathbb{R}^N} G^n(x;x^n)\,, \quad G^n(x;x^n) := g(x) + \langle \nabla f(x^n), x - x^n\rangle + \frac{1}{2\alpha_n}\|x - y^n\|^2_{A_n}\,, \end{aligned} \tag{7}$$

*where* $L_n > \sigma m_n$ *is determined such that*

$$f(x^{n+1}) \leq f(x^n) + \left\langle \nabla f(x^n), x^{n+1} - x^n \right\rangle + \frac{L_n}{2}\|x^{n+1} - x^n\|^2_{A_n} \tag{8}$$

*holds and* $\alpha_n, \beta_n$ *with* $\inf_n \alpha_n > 0$ *are chosen such that*

$$\delta_n^\sigma := \frac{1}{2}\left(\frac{1 + \sigma - \beta_n}{\alpha_n} - (L_n - \sigma m_n)\right) \quad \text{and} \quad \gamma_n := \delta_n^\sigma - \frac{\beta_n}{2\alpha_n} \tag{9}$$

*satisfy*

$$\inf_n \gamma_n > 0 \quad \text{and} \quad \delta_{n+1}^\sigma\|x^{n+1} - x^n\|^2_{A_{n+1}} \leq \delta_n^\sigma\|x^{n+1} - x^n\|^2_{A_n}\,, \tag{10}$$

*where* $m_n \in \mathbb{R}$ *denotes the semi-convexity modulus of* $g$ *w.r.t.* $A_n \in \mathbb{S}_{++}(N)$ *(if* $\sigma = 1$*).*

*Proof.* The bounds directly follow from $\inf_n \gamma_n > 0$. □

*Remark* 3. The minimization problem in (7) is equivalent to (constant terms are dropped)

$$\arg\min_{x\in\mathbb{R}^N} g(x) + \langle \nabla f(x^n), x - x^n\rangle - \frac{\beta_n}{\alpha_n}\left\langle x^n - x^{n-1}, x - x^n\right\rangle_{A_n} + \frac{1}{2\alpha_n}\|x - x^n\|^2_{A_n}\,. \tag{11}$$

The optimality condition of the minimization problem in (7) yields

$$0 \in \partial G^n(x;x^n) = \partial g(x) + \nabla f(x^n) + \frac{1}{\alpha_n}A_n(x - y^n)$$

and by plugging the expression for $y^n$ and a simple rearrangement, we obtain a necessary condition for $x^{n+1}$

$$x \in (\mathrm{id} + \alpha_n A_n^{-1}\partial g)^{-1}\left(x^n - \alpha_n A_n^{-1}\nabla f(x^n) + \beta_n(x^n - x^{n-1})\right)\,. \tag{12}$$

For a convex function $g$, inverting the expression $\mathrm{id} + \alpha_n A_n^{-1}\partial g$ yields a unique solution and the inclusion can be replaced by an equality. Here, the operator is still set-valued.

*Remark* 4.
- The assumption in (8) is satisfied for example, if $f$ has an $L$-Lipschitz continuous gradient with $A_n = \mathrm{id}$, or when a local estimate of the Lipschitz constant $L_n$ is known (also $A_n = \mathrm{id}$).
- Since $\nabla f$ is assumed to be Lipschitz continuous, given $A \in \mathbb{S}_{++}(N)$, we can always find $L$ such that $A_n$ can be "normalized" to $0 \preceq A \preceq \mathrm{id}$. Therefore, in practice the algorithm can be extended by a backtracking procedure for estimating $L_n$.

- The additional hyperparameters $\delta_n^\sigma$ and $\gamma_n$ can be seen as an disadvantage, however, actually, they allow for a constructive selection of the step size parameters (cf. [28]). For example in [12], such hyperparameters do not appear and only existence of parameters that satsify certain conditions can be guaranteed.
- Unlike in [28, 27], where the sequence $\delta_n$ is assumed to be stationary after a finite number of iterations to obtain the final convergence result, here, the restrictions for $\delta_n$ and $A_n$ are very loose: essentially boundedness is required.

As mentioned before, we want to take advantages out of $g$ being semi-convex. The next lemmas are essential for that.

**Lemma 13.** *Let $g$ be proper semi-convex with modulus $m \in \mathbb{R}$ with respect to the metric induced by $A \in \mathbb{S}_{++}(N)$, and fix $\tilde{x} \in \operatorname{dom} g$. Then, for any $\bar{x} \in \operatorname{dom} \partial g$ it holds that*

$$g(x) \geq g(\bar{x}) + \langle \bar{v}, x - \bar{x} \rangle + \frac{m}{2} \|x - \bar{x}\|_A^2 , \quad \forall x \in \mathbb{R}^N \text{ and } \bar{v} \in \partial g(\bar{x}) .$$

*Proof.* Apply the subgradient inequality to $\mathfrak{g}(x) := g(x) - \frac{m}{2}\|x - \tilde{x}\|_A^2$ around the point $\bar{x}$, i.e., it holds that

$$\mathfrak{g}(x) \geq \mathfrak{g}(\bar{x}) + \langle \bar{w}, x - \bar{x} \rangle , \quad \forall x \in \mathbb{R}^N \text{and } \bar{w} \in \partial \mathfrak{g}(\bar{x}) .$$

Note that $\bar{w}$ is an element from the (convex) subdifferential. Due to the smoothness of $\frac{m}{2}\|x - \tilde{x}\|_A^2$, we can use the summation rule for the limiting subdifferential to obtain

$$\partial \mathfrak{g}(\bar{x}) = \partial \left( g - \frac{m}{2} \| \cdot - \tilde{x}\|_2^2 \right) (\bar{x}) = \partial g(\bar{x}) - mA(\bar{x} - \tilde{x}) ,$$

and, therefore, replacing $\bar{w}$ by $\bar{v} - mA(\bar{x} - \tilde{x})$ with $\bar{v} \in \partial g(\bar{x})$ in the subgradient inequality above, we obtain after using

$$2 \langle \bar{x} - \tilde{x}, x - \bar{x} \rangle_A = \|x - \tilde{x}\|_A^2 - \|\bar{x} - \tilde{x}\|_A^2 - \|x - \bar{x}\|_A^2$$

that the following inequality holds

$$\mathfrak{g}(x) + \frac{m}{2}\|x - \tilde{x}\|_A^2 \geq \mathfrak{g}(\bar{x}) + \frac{m}{2}\|\bar{x} - \tilde{x}\|_A^2 + \frac{m}{2}\|x - \bar{x}\|_A^2 + \langle \bar{v}, x - \bar{x} \rangle , \quad \forall x \in \mathbb{R}^N \text{ and } \bar{v} \in \partial g(\bar{x}) ,$$

which implies the statement. □

**Lemma 14.** *Let $\sigma = 1$ if $g$ is proper semi-convex with modulus $m \in \mathbb{R}$ with respect to the metric induced by $A \in \mathbb{S}_{++}(N)$ and $\sigma = 0$ otherwise. Then it*

$$G^n(x^{n+1}; x^n) + \frac{\sigma}{2} \left( m + \frac{1}{\alpha_n} \right) \|x^{n+1} - x^n\|_A^2 \leq G^n(x^n; x^n) . \tag{13}$$

*Proof.* Apply Lemma 13 with $x = x^n$ and $\bar{x} = x^{n+1}$ to the function $x \mapsto G^n(x; x^n)$ from (7), which is semi-convex with modulus $\sigma\,(m + \frac{1}{\alpha_n})$ with respect to the metric induced by $A$. □

**Verification of Assumption 5.** We define the proper lower semi-continuous function

$$F \colon \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{N \times N} \times \mathbb{R} \to \overline{\mathbb{R}} \quad \text{given by} \quad F(x, y, A, \delta) := H_{(\delta, A)}(x, y) := h(x) + \delta \|x - y\|_A^2$$

for some $A \in \mathbb{S}_{++}(N)$ and $\delta \in \mathbb{R}$. Regarding the variables in Assumption 5, the $u$-component of $F$ is treated as $u = (A, \delta)$, which allows the function $F$ to change depending on the metric $A$ and another parameter $\delta$. Convergence will be derived for the $x$ and $y$ variables only.

The following proposition verifies (H1), with $d_n = \|x^n - x^{n-1}\|_2$ and $a_n = \gamma_n$.

**Proposition 15** (Descent property)**.** *Let the variables and parameters be given as in Algorithm 1. Then, it holds that*

$$H_{(\delta_n^\sigma, A_n)}(x^{n+1}, x^n) \le H_{(\delta_n^\sigma, A_n)}(x^n, x^{n-1}) - \gamma_n \varsigma(A_n) \|x^n - x^{n-1}\|_2^2 \,, \tag{14}$$

*and the sequence* $(H_{(\delta_n^\sigma, A_n)}(x^n, x^{n-1}))_{n\in\mathbb{N}}$ *is monotonically decreasing.*

*Proof.* Combining (7) (in the equivalent form (11)) with (8) and (13) yields

$$\begin{aligned} f(x^{n+1}) + g(x^{n+1}) + \frac{\sigma}{2}\left(m + \frac{1}{\alpha_n}\right) \|x^{n+1} - x^n\|_{A_n}^2 & \\ \le f(x^n) + \left\langle \nabla f(x^n), x^{n+1} - x^n \right\rangle + \frac{L_n}{2} \|x^{n+1} - x^n\|_{A_n}^2 & \\ + g(x^n) - \left\langle \nabla f(x^n), x^{n+1} - x^n \right\rangle + \frac{\beta_n}{\alpha_n} \left\langle x^{n+1} - x^n, x^n - x^{n-1} \right\rangle_{A_n} - \frac{1}{2\alpha_n} \|x^{n+1} - y^n\|_{A_n}^2 & \\ = f(x^n) + g(x^n) + \frac{\beta_n}{\alpha_n} \left\langle x^{n+1} - x^n, x^n - x^{n-1} \right\rangle + \Big(\frac{L_n}{2} - \frac{1}{2\alpha_n}\Big) \|x^{n+1} - x^n\|_{A_n}^2 & \end{aligned}$$

and using $\langle a, b\rangle_M \le \frac{1}{2}(\|a\|_M^2 + \|b\|_M^2)$ for any $a, b \in \mathbb{R}^N$ and $M \in \mathbb{S}_{++}(N)$ implies the following inequality

$$h(x^{n+1}) \le h(x^n) + \frac{\beta_n}{2\alpha_n} \|x^n - x^{n-1}\|_{A_n}^2 - \frac{1}{2}\left(\frac{1 + \sigma - \beta_n}{\alpha_n} - (L_n - \sigma m)\right) \|x^{n+1} - x^n\|_{A_n}^2$$

and rearranging terms yields

$$h(x^{n+1}) + \delta_n^\sigma \|x^{n+1} - x^n\|_{A_n}^2 \le h(x^n) + \delta_n^\sigma \|x^n - x^{n-1}\|_{A_n}^2 - (\delta_n^\sigma - \frac{\beta_n}{2\alpha_n}) \|x^n - x^{n-1}\|_{A_n}^2 \,.$$

□

The parametrization of the step sizes is chosen as in [27] (see [27, Lemma 6.3] for well-definedness of the parameters.) Therefore, we obtain the same step size restrictions here, but with the flexibility to change the metric in each iteration.

*Remark* 5. The proof shows that instead of (11) we could also consider

$$\arg\min_{x\in\mathbb{R}^N}\ g(x) + \langle \nabla f(x^n), x - x^n\rangle - \frac{\beta_n}{\alpha_n} \left\langle x^n - x^{n-1}, x - x^n \right\rangle + \frac{1}{2\alpha_n} \|x - x^n\|_{A_n}^2 \,, \tag{15}$$

which yields a slightly different algorithm, but step size restrictions are the same.

Next, we prove the relative error condition (Assumption (H2)) with $b_n = 1$ and $\varepsilon_n = 0$. First, we derive a bound on the (limiting) subgradient of the function $h$ and then for the function $F$.

**Lemma 16.** *Let the variables and parameters be given as in Algorithm 1. Then, there exists* $b > 0$ *such that*

$$\|\partial h(x^{n+1})\|_- \le \frac{b}{2} \left( \|x^{n+1} - x^n\|_2 + \|x^n - x^{n-1}\|_2 \right) \,.$$

*Proof.* (12) can be used to specify an element from $\partial g(x^{n+1})$, namely

$$A_n \frac{x^n - x^{n+1}}{\alpha_n} - \nabla f(x^n) + \frac{\beta_n}{\alpha_n} A_n (x^n - x^{n-1}) \in \partial g(x^{n+1}) \,,$$

which implies

$$\|\partial h(x^{n+1})\|_- = \|\nabla f(x^{n+1}) + \partial g(x^{n+1})\|_- \le \left(\frac{\|A_n\|}{\alpha_n} + L\right) \|x^{n+1} - x^n\|_2 + \frac{\beta_n}{\alpha_n} \|A_n\| \|x^n - x^{n-1}\|_2$$

and thus, using the Lipschitz continuity of $\nabla f$ and $A \preceq \mathrm{id}$, the statement. □

**Proposition 17.** *Let the variables and parameters be given as in Algorithm 1. Then, there exists $b > 0$ such that*

$$\|\partial F(x^{n+1}, x^n, A_{n+1}, \delta^\sigma_{n+1})\|_- \le \frac{b}{2}\left(\|x^{n+1} - x^n\|_2 + \|x^n - x^{n-1}\|_2\right) .$$

*Proof.* Thanks to summation rule of the limiting subdifferential for the sum of $(x, y, A, \delta) \mapsto h(x)$ and the smooth function $(x, y, A, \delta) \mapsto \delta\|x^{n+1} - x^n\|_A^2$, we can compute the limiting subdifferential by estimating the partial derivatives. We obtain

$$\partial_x F(x, y, A, \delta) = \partial h(x) + 2\delta A(x - y) , \qquad \partial_y F(x, y, A, \delta) = \nabla_y F(x, y, A, \delta) = -2A\delta A(x - y) \tag{16}$$

$$\partial_A F(x, y, A, \delta) = \nabla_A F(x, y, A, \delta) = \delta(x - y) \otimes (x - y) , \qquad \partial_\delta F(x, y, A, \delta) = \nabla_\delta F(x, y, A, \delta) = \|x - y\|_A^2 . \tag{17}$$

In order to verify (H2), let $F^{n+1} := F(x^{n+1}, x^n, A_{n+1}, \delta^\sigma_{n+1})$ and we use $\|w^{n+1}\|_2 \le \|w_x^{n+1}\|_2 + \|w_y^{n+1}\|_2 + \|w_A^{n+1}\|_2 + \|w_\delta^{n+1}\|_2$ where $w_x^{n+1} \in \partial_x F^{n+1}$, $w_y^{n+1} = \nabla_y F^{n+1}$, $w_A^{n+1} = \nabla_A F^{n+1}$, and $w_\delta^{n+1} = \nabla_\delta F^{n+1}$. We obtain the relative error bound (H2) using $A_{n+1} \preceq \mathrm{id}$ and boundedness of $\delta^\sigma_{n+1}$, Lemma 16, and the fact that for a sequence $r_n \to 0$ for some $n_0 \in \mathbb{N}$ it holds that $r_n^2 \le r_n$ for all $n \ge n_0$. In detail, we use

$$\|w_A^{n+1}\|_2 \le \delta^\sigma_{n+1} \sum_{i,j} |x_i^{n+1} - x_i^n| \cdot |x_j^{n+1} - x_j^n| \le c \sum_{i,j} |x_j^{n+1} - x_j^n| \le cc' \sum_i \|x^{n+1} - x^n\|_2 \le cc'c''\|x^{n+1} - x^n\|_2 ,$$

where $c$ is the maximal (over the coordinates $i$) bound for the converging sequences $|x_i^{n+1} - x_i^n| \to 0$ as $n \to \infty$, the dimensionally dependent constant $c' = \sqrt{N}$ provides the norm equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$, and $c'' = N$ simplifies the summation. ☐

Eventually, we verify the continuity Condition (H3).

**Proposition 18.** *Let the variables and parameters be given as in Algorithm 1. Then, there exists a subsequence $((x^{n_j+1}, x^{n_j}, A_{n_j}, \delta^\sigma_{n_j}))_{j\in\mathbb{N}}$ that $F$-converges to a point $(x^*, x^*, A_*, \delta^\sigma_*)$. Moreover, any convergent subsequence is $F$-convergent.*

*Proof.* The existence is of a converging subsequence is immediate from $(H_{(\delta^\sigma_n, A_n)}(x^n, x^{n-1}))_{n\in\mathbb{N}}$ begin non-increasing and the coercivity of $h$, which implies compact lower level sets for $H$, and the boundedness of $(A_n)_{n\in\mathbb{N}}$ and $(\delta^\sigma_n)_{n\in\mathbb{N}}$.

Now, let $(x^{n_j+1}, x^{n_j}, A_{n_j}, \delta^\sigma_{n_j})$ be a subsequence converging to some $(x^*, x^*, A_*, \delta^\sigma_*)$.

The continuity statement follows ($G^n(x^{n+1}; x^n) \le G^n(x; x^n)$ for all $x \in \mathbb{R}^N$ from (7)) from

$$g(x^{n_j+1}) + \langle \nabla f(x^{n_j}), x^{n_j+1} - x^{n_j}\rangle + \frac{1}{2\alpha_{n_j}}\|x^{n_j+1} - y^{n_j}\|_{A_{n_j}}^2 \le g(x) + \langle \nabla f(x^{n_j}), x - x^{n_j}\rangle + \frac{1}{2\alpha_{n_j}}\|x - y^{n_j}\|_{A_{n_j}}^2 .$$

Due to Lemma 6(iii) $\|x^{n_j+1} - x^{n_j}\| \to 0$, hence $\|y^{n_j} - x^{n_j}\| \to 0$, which shows that $y^{n_j} \to x^*$, as $j \to \infty$. Therefore considering the limit superior of $j$ to infinity of both sides of the inequality shows that $\limsup_{j\to\infty} g(x^{n_j+1}) \le g(x^*)$, which combined with the lower semi-continuity of $g$ and differentiability of $f$ implies $\lim_{j\to\infty} g(x^{n_j+1}) = g(x^*)$, and thus the statement follows. ☐

Using the results that we just derived, we can prove convergence of the variable metric iPiano method (Algorithm 1) to a critical point. Unlike the abstract convergence theorems in [5, 16, 28], the finite length property is derived for the coordinates from a subspace only, which allows for a lot of flexibility. Critical points are characterized in the proof of Proposition 17 (see (16)), where zero in the partial subdifferential (actually the partial derivative) with respect to $y$, $A$, or $\delta$ implies $x = y$ without imposing conditions on the $\delta$- or $A$-coordinate. Thus, the we have

$$0 \in \partial F(x, y, A, \delta) \Leftrightarrow \Big(0 \in \partial h(x) \times 0_y \times 0_A \times 0_\delta \text{ and } x = y\Big) \Leftrightarrow \Big(0 \in \partial h(x) \text{ and } x = y\Big) ,$$

where we indicate the size of the zero variables by the respective coordinate variable. As a consequence, $0 \in F(x^*, x^*, \delta, A) \Leftrightarrow 0 \in \partial h(x^*)$. These considerations lead to the following convergence theorem.

**Theorem 19.** *Suppose $F$ is a proper lower semi-continuous Kurdyka–Lojasiewicz function that is bounded from below. Let $(x^n)_{n\in\mathbb{N}}$ be generated by Algorithm 1 with valid variables and parameters as in the description of this algorithm. Then, the sequence $(x^n)_{n\in\mathbb{N}}$ satisfies*

$$\sum_{k=0}^{\infty} \|x^{k+1} - x^k\|_2 < +\infty\,, \tag{18}$$

*and $(x^n)_{n\in\mathbb{N}}$ converges to a critical point of* (6).

*Proof.* Verify the condition in Assumption 5 and apply Theorem 11. Set $d_n = \|x^n - x^{n-1}\|_2$, $a_n = \gamma_n \varsigma(A_n)$, $b_n = 1$, $\varepsilon_n = 0$, then (H1), (H2), and (H3) are proved in Propositions 15, 17, and 18, and (H4) is immediate from the bounds on the parameters. □

*Remark* 6. Thanks to [7, 8] the KL property holds for proper lower semi-continuous functions that are definable in an o-minimal structure, e.g., semi-algrabraic functions. Since o-minimal structures are stable under various operations, $F$ is a KL function if $h$ is definable in an o-minimal structure. Therefore, Theorem 19 can be applied to, for instance, a proper lower semi-continuous semi-algebraic function $h$ in (6).

## 5 Block coordinate variable metric iPiano

We consider a structured nonsmooth, nonconvex optimization problem with a proper lower semi-continuous extended valued function $h\colon \mathbb{R}^N \to \overline{\mathbb{R}}$, $N \geq 1$:

$$\min_{\mathbf{x}\in\mathbb{R}^N} \; h(\mathbf{x}) = \min_{\mathbf{x}\in\mathbb{R}^N} \; f(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_J) + \sum_{i=1}^{J} g_i(\mathbf{x}_i)\,, \tag{19}$$

where the $N$ dimensions are partitioned into $J$ blocks of (possibly different dimensions) $(N_1, \ldots, N_J)$, i.e., $\mathbf{x} \in \mathbb{R}^N$ can be decomposed as $\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_J)$. The function $f\colon \mathbb{R}^N \to \mathbb{R}$ is assumed to be block $C^1$-smooth (possibly nonconvex) with block Lipschitz continuous gradient on $\operatorname{dom} g_1 \times \operatorname{dom} g_2 \times \ldots \times \operatorname{dom} g_J$, i.e., $\mathbf{x}_i \mapsto \nabla_{\mathbf{x}_i} f(\mathbf{x}_1, \ldots, \mathbf{x}_i, \ldots, \mathbf{x}_J)$ is Lipschitz continuous. Further, let the function $g_i\colon \mathbb{R}^{N_i} \to \overline{\mathbb{R}}$ be simple (possibly nonsmooth and nonconvex) and prox-bounded. We require $h$ to be coercive and bounded from below by some value $\underline{h} > -\infty$.

Working with block algorithms can be simplified by an appropriate notation, which we introduce now. We denote by $\mathbf{x}_{\bar{i}} := (\mathbf{x}_1, \ldots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_J)$ the vector containing all blocks but the $i$th one.

The following Algorithm 2 is a straight forward extension of Algorithm 1 to problems of class (19) with a block coordinate structure. In each iteration, the algorithm applies one iteration of iPiano to the problem restricted to a certain block. The formulation of the algorithm allows blocks to be updated in an almost arbitrary order. In the end, the only restriction is that each block must be updated infinitely often.

We seek for a critical point $\mathbf{x}^* \in \operatorname{dom} h$ of $h$, which in this case is characterized by

$$-\nabla f(\mathbf{x}) \in \partial g_1(\mathbf{x}_1) \times \partial g_2(\mathbf{x}_2) \times \ldots \times \partial g_J(\mathbf{x}_J)\,.$$

In fact if we apply Algorithm 2 to (6) from the preceding section (i.e. $J = 1$), we recover the variable metric iPiano algorithm (Algorithm 1). For $\beta_{n,i} = 0$ for all $n \in \mathbb{N}$ and $i \in \{1, \ldots, J\}$, the algorithm is known as Block Coordinate Variable Metric Forward-Backward (BC-VMFB) algorithm [14]. If, additionally $A_{n,i} = \mathrm{id}$ for all $n$ and $i$, the algorithm is referred to as Proximal Alternating Linearized Minimization (PALM) [10].

**Verification of Assumption 5.** In order to prove convergence of this algorithm, we can make use of the results of the preceding section for the variable metric iPiano algorithm. We consider a function

$$F\colon \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^{N_1\times N_1} \times \ldots \times \mathbb{R}^{N_J\times N_J} \times \mathbb{R}^J \to \overline{\mathbb{R}}$$

**Algorithm 2.** ***Block coordinate variable metric iPiano***

- ***Parameter***: *Let for all* $i \in \{1, \ldots, J\}$
  - $(\alpha_{n,i})_{n\in\mathbb{N}}$ *be a sequence of positive step size parameters,*
  - $(\beta_{n,i})_{n\in\mathbb{N}}$ *be a sequence of non-negative parameters, and*
  - $(A_{n,i})_{n\in\mathbb{N}}$ *be a sequence of matrices* $A_{n,i} \in \mathbb{S}_{++}(N_i)$ *such that* $A_{n,i} \preceq \mathrm{id}$ *and* $\inf_{n,i} \varsigma(A_{n,i}) > 0$.
  - *Let* $\sigma_i = 1$ *if* $g_i$ *is semi-convex and* $\sigma_i = 0$ *otherwise.*
- ***Initialization***: *Choose a starting point* $x^0 \in \mathrm{dom}\, h$ *and set* $x^{-1} = x^0$.
- ***Iterations*** $(n \geq 0)$: *Update: Select* $j_n \in \{1, \ldots, J\}$ *and compute*

$$\begin{aligned}
\mathbf{y}^n_{j_n} &= \mathbf{x}^n_{j_n} + \beta_{n,j_n}(\mathbf{x}^n_{j_n} - \mathbf{x}^{n-1}_{j_n}) \\
\mathbf{x}^{n+1}_{j_n} &\in \arg\min_{x\in\mathbb{R}^{N_{j_n}}} G^{j_n}(x; \mathbf{x}^n_{j_n}) \\
&\qquad G^{j_n}(x; \mathbf{x}^n_{j_n}) := g_{j_n}(x) + \left\langle \nabla_{\mathbf{x}_{j_n}} f(\mathbf{x}^n), x - \mathbf{x}^n_{j_n} \right\rangle + \frac{1}{2\alpha_{n,j_n}} \|x - \mathbf{y}^n_{j_n}\|^2_{A_{n,j_n}} \qquad (20) \\
\mathbf{x}^{n+1}_{\bar{j}_n} &= \mathbf{x}^n_{\bar{j}_n} \\
\mathbf{x}^n_{\bar{j}_n} &= \mathbf{x}^{n-1}_{\bar{j}_n},
\end{aligned}$$

*where* $L_n > \sigma m_n$ *is determined such that*

$$f(\mathbf{x}^{n+1}) \leq f(\mathbf{x}^n) + \left\langle \nabla_{\mathbf{x}_{j_n}} f(\mathbf{x}^n), \mathbf{x}^{n+1}_{j_n} - \mathbf{x}^n_{j_n} \right\rangle + \frac{L_n}{2} \|\mathbf{x}^{n+1}_{j_n} - \mathbf{x}^n_{j_n}\|^2_{A_{n,j_n}} \qquad (21)$$

*holds and* $\alpha_{n,j_n}, \beta_{n,j_n}$ *with* $\inf_{n,j} \alpha_{n,j} > 0$ *are chosen such that*

$$\delta^{\sigma_{j_n}}_{n,j_n} := \frac{1}{2}\left(\frac{1 + \sigma_{j_n} - \beta_{n,j_n}}{\alpha_{n,j_n}} - (L_n - \sigma_{j_n} m_n)\right) \quad \text{and} \quad \gamma_{n,j_n} := \delta^{\sigma_{j_n}}_{n,j_n} - \frac{\beta_{n,j_n}}{2\alpha_{n,j_n}} \qquad (22)$$

*satisfy*

$$\inf_{n,j} \gamma_{n,j} > 0 \quad \text{and} \quad \delta^{\sigma_{j_n}}_{n+1,j_n} \|\mathbf{x}^{n+1}_{j_n} - \mathbf{x}^n_{j_n}\|^2_{A_{n+1,j_n}} \leq \delta^{\sigma_{j_n}}_{n,j_n} \|\mathbf{x}^{n+1}_{j_n} - \mathbf{x}^n_{j_n}\|^2_{A_{n,j_n}}, \qquad (23)$$

*where* $m_n \in \mathbb{R}$ *denotes the semi-convexity modulus of* $g_{j_n}$ *w.r.t.* $A_{j_n} \in \mathbb{S}_{++}(N_{j_n})$ *(if* $\sigma_{j_n} = 1$*). Set* $A_{n+1,\bar{j}_n} = A_{n,\bar{j}_n}$, $\delta^{\sigma_{j_n}}_{n+1,\bar{j}_n} = \delta^{\sigma_{j_n}}_{n,\bar{j}_n}$.

given by (set $\mathbf{A} := (A_1, \ldots, A_J)$, $A_i \in \mathbb{R}^{N_i \times N_i}$, $\mathbf{\Delta} := (\delta_1, \ldots, \delta_J)$)

$$F(\mathbf{x}, \mathbf{y}, \mathbf{A}, \mathbf{\Delta}) = H_{\mathbf{\Delta},\mathbf{A}}(\mathbf{x}, \mathbf{y}) := h(\mathbf{x}) + \sum_{i=1}^{J} \delta_i \|\mathbf{x}_i - \mathbf{y}_i\|^2_{A_i}.$$

**Theorem 20.** *Suppose* $F$ *is a proper lower semi-continuous Kurdyka–Łojasiewicz function (e.g.* $h$ *is semi-algebraic; cf. Remark 6) that is bounded from below. Let* $(\mathbf{x}^n)_{n\in\mathbb{N}}$ *be generated by Algorithm 2 with valid variables and parameters as in the description of this algorithm. Assume that each block coordinate is updated infinitely often. Then, the sequence* $(\mathbf{x}^n)_{n\in\mathbb{N}}$ *satisfies*

$$\sum_{k=0}^{\infty} \|\mathbf{x}^{k+1} - \mathbf{x}^k\|_2 < +\infty, \qquad (24)$$

*and* $(\mathbf{x}^n)_{n\in\mathbb{N}}$ *converges to a critical point of* (19).

*Proof.* As the $n$th iteration of Algorithm 2 reads exactly the same as in Algorithm 1 but applied to the block coordinate $j_n$ only, we can directly apply Propositions 15, and obtain

$$H_{(\mathbf{\Delta}_n^\sigma,\mathbf{A}_n)}(\mathbf{x}^{n+1},\mathbf{x}^n) \le H_{(\delta_n^\sigma,A_n)}(\mathbf{x}^n,\mathbf{x}^{n-1}) - \gamma_{n,j_n}\varsigma(A_{n,j_n})\|\mathbf{x}_{j_n}^n - \mathbf{x}_{n_j}^{n-1}\|_2^2\,,$$

and the function $H$ is monotonically decreasing along the iterations, i.e., the parameters in the algorithm are chosen such that one step on an arbitrary block decreases the value of $H$ unless the block coordinate is already stationary.

Since the non-smooth part of the optimization problem (19) is additively separated the estimation of the subdifferential is simple as it reduces to the Cartesian product of the subdifferential with respect to each block. Therefore, Proposition 17 can be used analogously to deduce

$$\|\partial F(\mathbf{x}^{n+1},\mathbf{y}^{n+1},\mathbf{A}_{n+1},\mathbf{\Delta}_{n+1})\|_- \le \frac{b}{2}\left(\|\mathbf{x}_{j_n}^{n+1} - \mathbf{x}_{j_n}^n\|_2^2 + \|\mathbf{x}_{j_n}^n - \mathbf{x}_{j_n}^{n-1}\|_2^2\right)\,.$$

Under the assumption that each block is updated infinitely often, also the continuity results from Proposition 18 can be transferred easily to the setting of Algorithm 2, i.e., we can conclude any convergent subsequence of block coordinates actually $F$-converges to the limit point ($\lim_{k\to\infty} g_i(\mathbf{x}_i^{n_k}) = g_i(\mathbf{x}_i^*)$ for each block $i \in \{1,\dots,J\}$ and $f$ is continuous anyway).

Therefore, the conditions in Assumption 5 are verified by $d_n = \|\mathbf{x}_{j_n}^n - \mathbf{x}_{j_n}^{n-1}\|_2$, $a_n = \gamma_{n,j_n}\varsigma(A_{n,j_n})$, $b_n = 1$, and $\varepsilon_n = 0$. □

# 6 Numerical application

## 6.1 A Mumford–Shah-like problem

The continuous Mumford–Shah problem is given formally by

$$\min_{w,\Gamma} \frac{\lambda}{2}\int_\Omega |w - I|^2\,dx + \int_{\Omega\setminus\Gamma} |\nabla w|^2\,dx + \gamma|\Gamma|\,, \tag{25}$$

where $w\colon \Omega \to \mathbb{R}$ is an image on the image domain $\Omega \subset \mathbb{R}^2$ and $I\colon \Omega \to \mathbb{R}$ is a given noisy image, $|\Gamma|$ measures the length of the jump set $\Gamma$. Intuitively, a solution $w$ must be smooth except on a possible jump set $\Gamma$, and approximate $I$. The positive parameters $\lambda$ and $\gamma$ steer the importance of each term. In order to solve the problem, the jump set $\Gamma$ needs to represented with a mathematical object that is amenable for a numerical implementation.

Therefore, we consider the well-known Ambrosio–Tortorelli approximation [2] given by

$$\min_{w,z} \frac{\lambda}{2}\int_\Omega |w - I|^2\,dx + \int_\Omega z^2|\nabla w|^2\,dx + \gamma\int_\Omega \varepsilon|\nabla z|^2 + \frac{(z-1)^2}{4\varepsilon}\,dx\,, \tag{26}$$

where $\varepsilon > 0$ is a fixed parameter and $z\colon \Omega \to [0,1]$ is a (soft) edge indicator function, also called a phase-field. The last integral is shown to Gamma-converge to the length of the jump set of (25) as $\varepsilon \to 0$.

In this section, we solve a slight variation of this problem. Instead of an image denoising model we are interested in an inpainting problem (as shown in Figure 2), which is usually more difficult. In image inpainting, the true information about the original image is only given on a subset $[c = 1]$ of the image domain (black pixels in Figure 2(b)), where $c\colon \Omega \to \{0,1\}$—the original image $I$ is unknown on $[c = 0]$ (white part Figure 2(b)). In [17], the idea of image inpainting is pushed to a limit and used for PDE-based image compression, i.e., the inpainting mask $[c = 1]$ is a small subset of $\Omega$. Usually a simple PDE is used for reconstructing the original image based on its gray values given only on mask points, for instance linear diffusion in [23] (result given in Figure 2(d)). When the inpainting mask is optimized, linear diffusion based inpainting is shown to be competetive with JPEG and sometimes with JPEG2000. Therefore using

(a) original image $I$ (b) mask $c$ (90% unknown) (c) inpainting $w$ using (27) (d) linear diffusion inpainting

Figure 2: Example for image inpainting/compression. The gray values of the original image (a) are stored only at the mask points (b), where known values are black $[c = 1]$ and unknown ones are white $[c = 0]$. Based on 10% known gray values the original image is reconstructed in (c) with the Ambrosio–Tortorelli inpainting (27) that we evaluate algorithmically in this paper, and in (d) with a simple linear diffusion model [23] which arises as a special case of (27) when the edge set $z$ is fixed to 1 everywhere on the image domain $\Omega$.

a more general inpainting model combined with an optimized inpainting mask is expected to improve this performance. We consider the model

$$\begin{aligned} &\min_{w,z} \int_\Omega z^2 |\nabla w|^2 \, dx + \gamma \int_\Omega \varepsilon |\nabla z|^2 + \frac{(z-1)^2}{4\varepsilon} \, dx \\ &s.t.\ w(x) = I(x)\,, \quad \forall x \in [c = 1]\,, \end{aligned} \tag{27}$$

which extends the linear diffusion model by optimizing for an additional edge set $z$. The linear diffusion model is recovered when fixing $z = 1$ on $\Omega$. Since we want to evaluate our algorithms, we neglect the development made for finding an optimal inpainting mask and generate the mask by randomly selecting 10% as known pixels.

From now on, we discretize the problem and with a slight abuse of notation we use the same symbols to denote the discrete counterparts of the above introduced variables: $I \in \mathbb{R}^N$ is the (vectorized[1]) original image, $c \in \mathbb{R}^N$ is the (inpainting) mask, $w \in \mathbb{R}^N$ is the optimization variable (representing a vectorized image), and $z \in [0,1]^N$ represents the jump (or edge) set of (25). The continuous gradient $\nabla$ is replaced by a discrete derivative operator $D \in \mathbb{R}^{2N\times N}$ that implements forward differences in horizontal $D_1 \in \mathbb{R}^{N\times N}$ and vertical direction $D_2 \in \mathbb{R}^{N\times N}$ with homogeneous boundary conditions, i.e., forward differences across the image boundary are set to 0. Our discretized model of (27) reads

$$\begin{aligned} &\min_{w,z} \frac{1}{2}\| \operatorname{diag}(z)(D_1 w)\|_2^2 + \frac{1}{2}\| \operatorname{diag}(z)(D_2 w)\|_2^2 + \frac{\gamma\varepsilon}{2}\|Dz\|_2^2 + \frac{\gamma}{4\varepsilon}\|z - 1\|_2^2 \\ &s.t.\ w_i = I_i\,, \quad \forall i \in \{1, \dots, N\} \text{ with } c_i = 1\,, \end{aligned} \tag{28}$$

where $\operatorname{diag}\colon \mathbb{R}^N \to \mathbb{R}^{N\times N}$ puts a vector on the diagonal of a matrix. Figure 4 shows the input data, the reconstructed image, and the reconstructed edge set, for $\varepsilon = 0.1$ and $\gamma = 1/400$ and the number of pixel $N = 551 \cdot 414 = 228114$.

In the following, we evaluate several algorithms that use a variable metric. Let

$$g_1(w) := \delta_X(w) \text{ with } X := \{w \in \mathbb{R}^N \,|\, w_i = I_i \text{ if } c_i = 1\}\,, \qquad g_2(z) := \frac{\gamma}{4\varepsilon}\|z - 1\|_2^2$$

$$f(w, z) := \frac{1}{2}\left(\| \operatorname{diag}(z)(D_1 w)\|_2^2 + \| \operatorname{diag}(z)(D_2 w)\|_2^2 + \gamma\varepsilon\|Dz\|_2^2\right)\,.$$

[1]The columns of the image are stacked to a long vector.

We can apply iPiano to (6) with $x = (w, z)$ and $g(x) = (g_1(w), g_2(z))$, or block coordinate iPiano to (19) with $\mathbf{x}_1 = w$ and $\mathbf{x}_2 = z$.

In order to determine a suitable metric, we first compute the derivatives of $f$

$$\begin{aligned}\nabla_w f(w,z) &= \left(D_1^\top \operatorname{diag}(z^2) D_1 + D_2^\top \operatorname{diag}(z^2) D_2\right) w \\ \nabla_z f(w,z) &= \left(\operatorname{diag}((D_1 w)^2) + \operatorname{diag}((D_2 w)^2) + \gamma\varepsilon D^\top D\right) z\,,\end{aligned}$$

where the squares are to be understood coordinate-wise. A feasible metric for block coordinate variable metric iPiano (`BC-VM-iPiano`) must satisfy (21). Therefore, for the $w$-update step ($z$ is fixed), we require $A_{n,w}$ (the metric w.r.t. the block of $w$ coordinates) to satisfy

$$\langle \nabla_w f(w,z) - \nabla_w f(w',z) - A_{n,w}(w - w'), w - w' \rangle \leq 0$$

for all $w, w'$, which is achieved, for example, by a diagonal matrix $A_{n,w}$ given by

$$(A_{n,w})_{i,i} = \sum_{j=1}^{N} |\left(D_1^\top \operatorname{diag}(z^2) D_1 + D_2^\top \operatorname{diag}(z^2) D_2\right)_{i,j}| \tag{29}$$

for all $i \in \{1, \ldots, N\}$. For the $z$-update ($w$ is fixed), analogously, we require $A_{n,z}$ (the metric w.r.t. the block of $w$ coordinates) to satisfy

$$\langle \nabla_w f(w,z) - \nabla_w f(w,z') - A_{n,z}(z - z'), z - z' \rangle \leq 0$$

for all $z, z'$, which is achieved, for example, by a diagonal matrix $A_{n,z}$ given by

$$(A_{n,z})_{i,i} = \sum_{j=1}^{N} |\left(\operatorname{diag}((D_1 w)^2) + \operatorname{diag}((D_2 w)^2) + \gamma\varepsilon D^\top D\right)_{i,j}| \tag{30}$$

for all $i \in \{1, \ldots, N\}$. Note that compared to (21) the metric contains the scaling $L_{n,w}$ and $L_{n,z}$, respectively. For constant step size schemes ($A_{n,w} = A_{n,z} = \mathrm{id}$) we use $L_w \leq 8$ and[2] $L_z \leq 2 + 8\gamma\varepsilon$.

Besides `BC-VM-iPiano`, we test forward–backward splitting (`FB`) with constant step size scheme $\alpha = 2/\max(L_w, L_z)$, block coordinate forward–backward splitting (`BC-FB`) with step sizes $\alpha_w = 2/L_w$ and $\alpha_z = 2/L_z$ (this method is also known as PALM [10]), variable metric forward–backward splitting (`BC-FB`) with the metric (29) and (30) as a composed diagonal matrix, block coordinate variable metric forward–backward splitting (`BC-VM-FB`) with the metric (29) and (30), iPiano (`iPiano`) with constant step size scheme $\alpha = 2(1-\beta)/\max(L_w, L_z)$, block coordinate iPiano (`BC-iPiano`) with constant step size scheme $\alpha_w = 2(1-\beta)/L_w$ and $\alpha_z = 2(1-\beta)/L_z$, variable metric iPiano (`VM-iPiano`) with the metric (29) and (30) as a composed diagonal matrix, and block coordinate variable metric iPiano (`BC-VM-iPiano`) with the metric (29) and (30). For all methods that incorporate an inertial parameter, it is set to $\beta = 0.7$.

The metric that is used for `BC-FB` and `VM-iPiano` is actually not feasible, as (29) and (30) are not sufficient to guarantee that the metric induces a quadratic majorizer to the function $f$ (cf. (8)). The gradient is not linear with respect to both coordinates. The gradient is linear only if one coordinate is fixed. Nevertheless, in our practical experiments, the methods converged. In future work, we want to analyze if this inaccuracy can be compensated by making use of relative error conditions, which are not yet incorporated into the algorithms.

We solve problem (28) with all methods up to 1000 iterations and define $E^*$ as the minimal objective value that is achieved among all methods. Let $E^0$ be the initial value. Figure 3 plots the decrease of the relative objective value $(E^n - E^*)/(E^0 - E^*)$ along the iterations $n$ on a logarithmic scale on both axes.

The performance of `FB` and `iPiano` are nearly identical as they do not explore the different scaling of $w$- and $z$-coordinates, unlike `BC-FB` and `BC-iPiano`. As both block coordinates seem to "live" on a different scale, block coordinate methods are favorable. However, as the immense performance speed up

[2] Note that $I$ is normalized to $[0, 1]$ and, thus, we observed that $w$ stays in $[0, 1]$ too. Therefore $(D_1 w)_i^2$ is in $[0, 1]$.

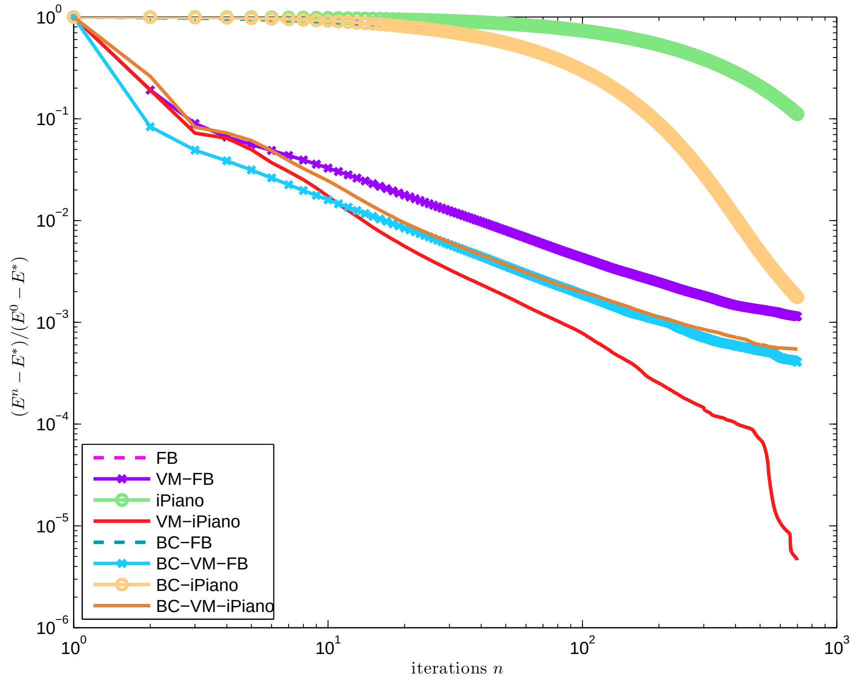


Figure 3: Number of iterations vs. relative objective value for solving (28). The performance is significantly improved for methods that take a variable metric into account. Intuitively, this means that the coordinates of the optimization variable are irregularly scaled along the iterations. The variable metric version of iPiano shows the best performance.

of the variable metric methods shows the irregular scaling happens to be present also among different $w$-coordinates, respectively, $z$-coordinates. Throughout the experiments, we have noticed that optimization problems where regularization (like smoothness between pixels) is important, inertial methods seem to perform slightly better in general. For this experiment variable metric iPiano shows the best performance and sets the value for $E^*$, the lowest objective value among all methods after 1000 iterations.

# 7 Conclusion

In this paper, we presented a convergence analysis for abstract inexact descent methods based on the KL-inequality that unifies and generalizes the analysis in Attouch et al. [5], Frankel et al. [16] and Ochs et al. [28]. The novel convergence theorem allows for more flexibility in the design of descent algorithms. More in detail, algorithms that imply a descent on a proper lower semi-continuous parametric function and satisfy a certain relative error condition are considered. The parametric function can be seen as an objective function that may vary along the iterations under mild restriction. The gained flexibility is used to formulate a variable metric version of iPiano (an inertial forward–backward splitting-like method). Moreover, thanks to a weakened contraction condition in the abstract convergence theorem, we obtain a block coordinate variable metric version of iPiano almost for free. Both algorithms are formulated such that they can be easily implemented with full control about the step size parameters. Finally, the algorithms are shown to perform well on the practical problem of image compression using a Mumford–Shah-like regularization.

Although the abstract convergence result provides the full flexibility in handling relative errors as in [16], we do not explore an inexact formulation of (block coordinate variable metric) iPiano in this paper, and postpone this research to future work.

(a) input to model (28) (b) inpainting $w$ using (28) (c) edges $z$ using (28)

Figure 4: Solution to Problem 28. (a) shows the inpainting mask from Figure 2(b) weighted with the gray values from Figure 2(a). (b) shows the solution image $w$ and (c) the solution edge set $z$ of (28). Although the model is non-convex, visually all algorithm resulted in the same solution. Figure 3 shows that the final objective values differ.

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