# Regularity and Scale-Space Properties of Fractional High Order Linear Filtering

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**Abstract.** We investigate the use of fractional powers of the Laplacian for signal and image simplification. We focus both on their corresponding variational techniques and parabolic pseudodifferential equations. We perform a detailed study of the regularisation properties of energy functionals, where the smoothness term consists of various linear combinations of fractional derivatives. The associated parabolic pseudodifferential equations with constant coefficients are providing the link to linear scale-space theory. These encompass the well-known $\alpha$-scale-spaces, even those with parameter values $\alpha > 1$ known to violate common maximum-minimum principles. Nevertheless, we show that it is possible to construct positivity-preserving combinations of high and low-order filters. Numerical experiments in this direction indicate that non-integral orders play an essential role in this construction. The paper reveals the close relation between continuous and semi-discrete filters, and by that helps to facilitate efficient implementations. In additional numerical experiments we compare the variance decay rates for white noise and edge signals through the action of different filter classes.

## 1 Introduction

Regularisation and diffusion filtering belong to the most frequently used and best studied methods in image processing. In addition to the well-known Gaussian scale-space [1, 2, 3, 4, 5], other linear scale-spaces enjoy a growing popularity. Already in the 1960's Iijima [6, 7] gave an axiomatic foundation of $\alpha$-scale-spaces with integer order using four axioms: linearity, translational invariance, scale invariance, and semigroup property. Later on a whole class of linear scale-spaces depending on a fractional order $\alpha > 0$ was axiomatically deduced (Pauwels et al. [8]). Duits et al. [9] further investigated the $\alpha$-scale-spaces where $\alpha \in (0, 1]$ can be interpreted as fractional power the Laplacian in a pseudodifferential equation creating the scale-space. The restriction on $\alpha$ comes from the demand of a

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maximum-minimum principle for the resulting filters. The most prominent representative of linear scale-spaces with fractional order is the Poisson scale-space by Felsberg and Sommer [10].

In our work we use fractional powers of the Laplacian not only in partial differential equations, but also in regularisation methods. Besides the scale-space properties we are especially interested in well-posedness and regularity properties. We see that variational methods allow it to prescribe a certain fractional regularity order for a given image where diffusion methods always yield arbitrary smooth solutions. In our experiments we propose a way to construct filters with maximum-minimum property which involve both high and low fractional derivative orders.

The paper is organised as follows. In Section 2 we introduce the basic notions related to fractional powers of the Laplacian. Section 3 presents fractional order regularisation as a first application of these notions. The corresponding diffusion equations are investigated in Section 4. Section 5 reformulates both approaches in a space-discrete framework directly leading to efficient implementations. Our numerical experiments in Section 6 especially are dedicated to the question of maximum-minimum property and variance decay. Section 7 concludes the paper.

## 2 Fractional Powers of the Laplacian

In order to present an elegant concept for fractional powers of the Laplacian, we have to introduce some basic notions first. First we consider the Fourier transform of a function $f \in L^1(\mathbb{R})$ pointwise defined by

$$\hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \exp(-ix\xi) dx$$

Let $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ denote the Fourier-Plancherel transform, i. e. the extension of the mapping $L^1(\mathbb{R}) \ni f \longrightarrow \hat{f}$ onto $L^2(\mathbb{R})$. It is well-known that $\mathcal{F}$ is isometric with respect to the norm in $L^2(\mathbb{R})$ (see [11] for details). Later on we will especially make use of the property

$$i^k \xi^k \mathcal{F} f = \mathcal{F}\left(\frac{d^k}{dx^k} f\right) \tag{1}$$

which builds the link between differentiation in the spatial domain and multiplication in the Fourier domain. For $f \in L^\infty(\mathbb{R})$ let $\mathcal{M}_f : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ denote the multiplication operator defined by $\mathcal{M}_f g := fg$. With this notation (1) reads as $\mathcal{M}_{(i\xi)^k} \mathcal{F} f = \mathcal{F}\left(\frac{d^k}{dx^k} f\right)$.

**Lemma 2.1.** *For $f, g \in L^\infty(\mathbb{R})$ the multiplication operator $\mathcal{M}_f$ is $L^2(\mathbb{R})$-continuous with $\|\mathcal{M}_f\| \leq \|f\|_\infty$. Further, $fg \in L^\infty(\mathbb{R})$ and $\mathcal{M}_f \mathcal{M}_g = \mathcal{M}_{fg}$.*

Following the notation in [12] we define the Sobolev space

$$H^s(\mathbb{R}) := \left\{ u \in L^2(\mathbb{R}) \mid \left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}) \right\} \tag{2}$$

of all functions in $L^2(\mathbb{R})$ and $s \in \mathbb{R}$. For $s \in \mathbb{N}$ functions in $H^s(\mathbb{R})$ are weakly differentiable up to the order $s$. From (1) we deduce the spectral decomposition of the Laplacian $-\frac{d^2}{dx^2} = \mathcal{F}^{-1}\mathcal{M}_{|\xi|^2}\mathcal{F}$ which allows us to define fractional powers

$$\mathcal{D}^{2\alpha} := \left(-\frac{d^2}{dx^2}\right)^{\alpha} = \mathcal{F}^{-1}\mathcal{M}_{|\xi|^{2\alpha}}\mathcal{F} \qquad (\alpha > 0) \tag{3}$$

as multiplication operators in the Fourier domain (see [13, 14] for further details).

**Lemma 2.2.** *Applying $\mathcal{D}^\alpha$ to functions in a certain Sobolev space reduces the order of differentiability by $\alpha$, i. e. $\mathcal{D}^\alpha : H^s(\mathbb{R}) \longrightarrow H^{s-\alpha}(\mathbb{R})$ for all $s \in \mathbb{R}$.*

In the next sections we are going to replace derivative operators in classical image processing approaches with operators of the type $\mathcal{D}^\alpha$ and investigate the properties of the resulting filter methods.

## 3 Regularisation with Fractional Derivative Orders

To extend linear regularisation to fractional derivative orders we consider the energy functional

$$\mathcal{E}(u) = \int_{\mathbb{R}} \left( (u-f)^2 + \sum_{k=1}^{m} \beta_k \left(\mathcal{D}^{\alpha_k} u\right)^2 \right) dx \tag{4}$$

with a linear combination of $m \in \mathbb{N}$ fractional derivatives of orders $\alpha_k > 0$ in the smoothness term and regularisation weights $\beta_k > 0$ for $k = 1, \ldots, m$, for short, $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^+_m$. For integer derivative orders $\alpha_k$, similar functionals have been considered in [15]. We assume that the signals $u$ and $f$ may only assume real values. With the Plancherel identity we can rewrite functional (4) in the Fourier domain as

$$\mathcal{E}(\hat{u}) = \int_{\mathbb{R}} \left( \left|\hat{u} - \hat{f}\right|^2 + \sum_{k=1}^{m} \beta_k \left|\xi^{\alpha_k} \hat{u}\right|^2 \right) d\xi \tag{5}$$

depending on the complex Fourier transform $\hat{u}$. A decomposition into the real and imaginary part shows that it is necessary for a minimiser $u$ to satisfy the Euler-Lagrange equation

$$\hat{u} - \hat{f} + \sum_{k=1}^{m} \beta_k |\xi|^{2\alpha_k} \hat{u} = 0 \quad \text{for all } \xi \in \mathbb{R} \ . \tag{6}$$

We deduce that the minimiser $u$ of the functional $\mathcal{E}$ has the Fourier transform

$$\hat{u} = \left(1 + \sum_{k=1}^{m} \beta_k |\xi|^{2\alpha_k}\right)^{-1} \hat{f} \quad \text{for all } \xi \in \mathbb{R} \ . \tag{7}$$

To obtain a regularised version of $f$ we transform this minimiser $\hat{u}$ in the spatial domain which motivates the following definition:

**Definition 3.1. (Fractional Order Regularisation)** *For* $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}_+^m$ *we denote the multipliers appearing in (7) with*

$$r_\beta^\alpha : \mathbb{R} \longrightarrow \mathbb{R}, \qquad r_\beta^\alpha(\xi) := \left(1 + \sum_{k=1}^m \beta_k |\xi|^{2\alpha_k}\right)^{-1} \tag{8}$$

*and use these functions to define the regularisation operators*

$$\mathcal{R}_\beta^\alpha : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R}), \qquad \mathcal{R}_\beta^\alpha = \mathcal{F}^{-1} \mathcal{M}_{r_\beta^\alpha} \mathcal{F} \ . \tag{9}$$

First we assure ourselves that the above definition leads to a continuous operator. Furthermore we give a measure for the increase of smoothness obtained by applying a regularisation operator of this class.

**Proposition 3.2. (Stability and Regularity of Regularisation)**

1. *The regularisation operator* $R_\beta^\alpha$ *is continuous with respect to the norm in* $L^2(\mathbb{R})$ *with* $\|R_\beta^\alpha\| \leq 1$.
2. *Regularisation increases the smoothness order by twice the minimal derivative order:*
   *For all* $s \in \mathbb{R}$ *it is* $\mathcal{R}_\beta^\alpha : H^s(\mathbb{R}) \longrightarrow H^{s+2\alpha^*}(\mathbb{R})$ *where* $\alpha^* := \min_{k=1,\ldots,m} \alpha_k$.

*Proof.* 1. The Fourier multipliers satisfy $0 \leq r_\beta^\alpha(\xi) \leq 1$ for all $\alpha, \beta \in \mathbb{R}_+^m$ and all $\xi \in \mathbb{R}$, i. e. $\|r_\beta^\alpha\|_{L^\infty(\mathbb{R})} \leq 1$. Lemma 2.1 then shows that $\left\|\mathcal{M}_{r_\beta^\alpha}\right\| \leq 1$ and

$$\|R_\beta^\alpha\| \leq \left\|\mathcal{F}^{-1}\right\| \left\|\mathcal{M}_{r_\beta^\alpha}\right\| \|\mathcal{F}\| \leq 1 \tag{10}$$

using the fact that the Fourier transform is $L^2$-isometric.

2. Fix $f \in H^s(\mathbb{R})$. First we note that

$$\left(1 + |\xi|^2\right)^{\frac{s}{2}} \hat{f} \in L^2(\mathbb{R}) \quad \Longleftrightarrow \quad |\xi|^s \hat{f} \in L^2(\mathbb{R}) \ . \tag{11}$$

Thus it follows that

$$\left(1 + \sum_{k=1}^m \beta_k |\xi|^{2\alpha_k}\right)^{-1} |\xi|^{s+2\alpha^*} \hat{f} \in L^2(\mathbb{R}) \tag{12}$$

which implies $\mathcal{R}_\beta^\alpha f \in H^{s+2\alpha^*}(\mathbb{R})$. □

For integer derivative orders a corresponding statement to the second part of the previous lemma can be found in [15]. As they state for integer orders, also fractional order regularisation is not a projection operator: Applying regularisation iteratively increases the smoothness in each step by twice the minimal derivative order $\alpha^*$. Starting with a function in $L^2(\mathbb{R})$ we now are able to reach a given degree of smoothness with linear regularisation. This smoothness property does not depend on the size of the regularisation weights $\beta_k > 0$. Two examples of

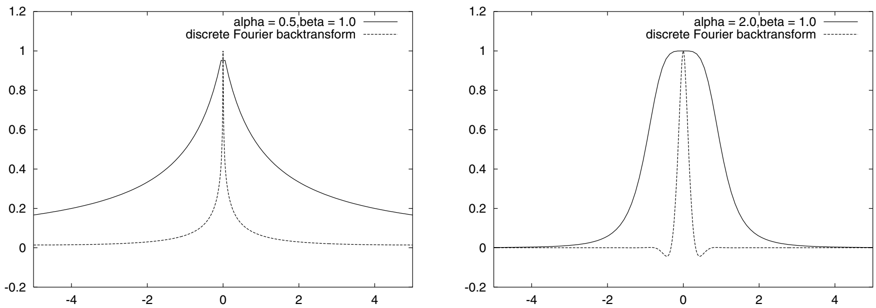


**Fig. 1.** Fourier multipliers and corresponding Fourier backtransforms for fractional order regularisation. Left: $\alpha = 0.5$, Right: $\alpha = 2.0$

the appearing Fourier multipliers are shown in Fig. 1. The multiplication in the Fourier domain can be related to convolution for which the corresponding kernels are also shown. The fact that the convolution kernel for $\alpha = 2.0$ reaches negative values indicates that the corresponding filter violates a maximum-minimum property. Besides its smoothing behaviour the linear filtering technique is also expected to satisfy some scale-space properties. We summarise these in the case of fractional order regularisation:

**Proposition 3.3. (Scale-Space Properties of Regularisation)** *The regularisation operators $\mathcal{R}^{\alpha}_{\beta}$ are linear, translational invariant and preserve the average grey value, i. e.* $\int_{\mathbb{R}} \left(\mathcal{R}^{\alpha}_{\beta} f\right)(x)dx = \int_{\mathbb{R}} f(x)dx.$

*Proof.* For the translational invariance we note that translations correspond to multiplications with phase factors $\exp(ic\xi)$ of absolute value one in the Fourier domain. Since the multipliers $r^{\alpha}_{\beta}$ only assume real values these do not affect the argument of the Fourier coefficients and thus do not interfere with the complex phase factors.

The average grey value can be expressed as $\hat{f}(0) = \int_{\mathbb{R}} f(x)\exp(-ix0)dx$. Since $r^{\alpha}_{\beta}(0) = 1$ for all $\alpha, \beta \in \mathbb{R}^m_+$, the average grey value remains unchanged by multiplication with $r^{\alpha}_{\beta}$ in the Fourier domain. □

## 4 Diffusion with Fractional Derivative Orders

The elliptic differential equations appearing in regularisation techniques are related to parabolic diffusion equations [16]. Now we investigate such parabolic equations involving a linear combination of different fractional powers of the Laplacian. To this purpose we choose fractional derivative orders $\alpha_1, \ldots, \alpha_m > 0$ and weight parameters $\lambda_1, \ldots, \lambda_m > 0$ and consider the linear pseudodifferential equation

$$\frac{\partial}{\partial t} u = -\sum_{k=1}^{m} \lambda_k \left( -\frac{\partial^2}{\partial x^2} \right)^{\alpha_k} u \ . \tag{13}$$

with initial condition $u(x,0) = f(x)$ for all $x \in \mathbb{R}$. In the Fourier domain (13) reads as $\frac{\partial}{\partial t}\hat{u} = -\sum_{k=1}^{m} \lambda_k |\xi|^{2\alpha_k} \hat{u}$. This is an ordinary differential equation with parameter $\xi$ and can be analytically solved by

$$\hat{u}(\xi,t) = \exp\left( -t \sum_{k=1}^{m} \lambda_k |\xi|^{2\alpha_k} \right) \hat{f} = \prod_{k=1}^{m} \exp\left( -t\lambda_k |\xi|^{2\alpha_k} \right) \hat{f} \ . \tag{14}$$

This formula expresses fractional order linear diffusion filtering as multiplication in the Fourier domain. The following definition uses its equivalence with convolution in the spatial domain.

**Definition 4.1. (Multipliers and Convolution Kernels for Diffusion)** *For the order $\alpha > 0$, the weight $\lambda > 0$ and the stopping time $t \geq 0$, we define the multiplier function*

$$G_\lambda^\alpha(\xi,t) := \exp\left( -t\lambda |\xi|^{2\alpha} \right) \quad \text{for all } \xi \in \mathbb{R} \ .$$

*We also define the convolution kernels appearing in linear filtering as the Fourier backtransform*

$$p_\lambda^\alpha(x,t) := \frac{1}{\sqrt{2\pi}} \mathcal{F}^{-1} \left( G_\lambda^\alpha(\cdot,t) \right)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left( -t\lambda |\xi|^{2\alpha} + ix\xi \right) d\xi \ .$$

We would like to mention that the convolution kernels $p_\lambda^\alpha(\cdot,t)$ were already discussed in [6] and [7] for $\alpha \in \mathbb{N}$. With this definition we are able to express the Fourier backtransform of the solution of (13) as convolution:

$$u(x,t) = \left( p_{\lambda_m}^{\alpha_m}(\cdot,t) * \ldots * p_{\lambda_1}^{\alpha_1}(\cdot,t) * f \right)(x) \ . \tag{15}$$

It is an interesting feature of (14) and (15) that one can successively add different derivative orders to the right-hand side of (13) and obtain the particular solution step by step by convolution with corresponding kernels. Figure 2 shows two Fourier multipliers for different diffusion orders and their associated convolution kernels obtained by numerical approximation.

As in the last section for regularisation, we also express fractional order diffusion as linear operator.

**Definition 4.2. (Fractional Order Diffusion)** *We choose fractional derivative orders $\alpha_1, \ldots, \alpha_m > 0$ and the corresponding weights $\lambda_1, \ldots, \lambda_m > 0$. For every $t \geq 0$ we define the linear filtering operator $\mathcal{T}_t : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ as*

$$\mathcal{T}_t f := \mathcal{F}^{-1} \mathcal{M}_{G_{\lambda_m}^{\alpha_m}(\cdot,t)} \cdot \ldots \cdot \mathcal{M}_{G_{\lambda_1}^{\alpha_1}(\cdot,t)} \mathcal{F} f \ . \tag{16}$$

With respect to stability and smoothness of the solutions, we see that these diffusion operators have very convenient properties.

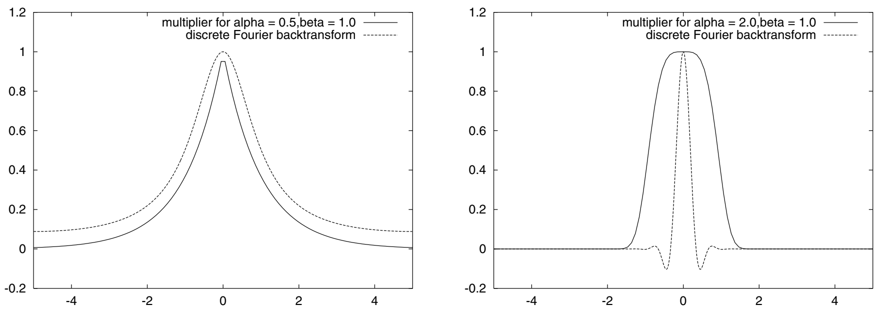


**Fig. 2.** Fourier multipliers and corresponding Fourier backtransforms for fractional order diffusion filtering. Left: $\alpha = 0.5$ (Poisson scale-space), Right: $\alpha = 2.0$

**Proposition 4.3. (Stability and Regularity of Diffusion)**

1. *For all* $t \geq 0$ *the operator* $\mathcal{T}_t$ *is continuous with respect to the norm in* $L^2(\mathbb{R})$ *with* $\|\mathcal{T}_t\| \leq 1$.
2. *For natural filter orders* $\alpha_1, \ldots, \alpha_m \in \mathbb{N}$ *it is* $T_t f \in C^\infty(\mathbb{R})$ *for initial data* $f \in L^2(\mathbb{R})$.
3. *For positive real filter orders* $\alpha_1, \ldots, \alpha_m > 0$ *we have* $T_t f \in H^k(\mathbb{R})$ *for arbitrary* $k \in \mathbb{N}$ *and initial data* $f \in L^2(\mathbb{R})$.

*Proof.* 1. $0 \leq G^\alpha_\lambda(\xi, t) \leq 1$ for all $t, \alpha, \lambda > 0$ and all $\xi \in \mathbb{R}$. An upper bound for the norm of $\mathcal{T}_t$ is given by

$$\|\mathcal{T}_t\| \leq \left\|\mathcal{F}^{-1}\right\| \left( \prod_{k=1}^{m} \left\| \mathcal{M}_{G^{\alpha_k}_{\lambda_k}(\cdot,t)} \right\| \right) \|\mathcal{F}\| \leq 1$$

with Lemma 2.1 (1.) and the fact that $\mathcal{F}$ is $L^2$-isometric.

2. For $\alpha \in \mathbb{N}$ the functions $G^\alpha_\lambda(\cdot, t)$ are in the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decreasing functions. Thus their Fourier backtransforms $p^\alpha_\lambda(\cdot, t)$ are also in $\mathcal{S}(\mathbb{R})$ and also the convolution kernel $p(\cdot, t) := p^{\alpha_m}_{\lambda_m}(\cdot, t) * \ldots * p^{\alpha_1}_{\lambda_1}(\cdot, t)$ appearing in linear filtering in the spatial domain. We see that the derivatives of $T_t f$ exist with

$$\frac{d^k}{dx^k} T_t f = \int_{\mathbb{R}} \frac{\partial^k}{\partial x^k} p(x-y) f(y) dx \ .$$

3. We note that $\lim\limits_{x \longrightarrow \infty} x^k \exp(-x^\alpha) = 0$ for all $k \in \mathbb{N}$. Thus we have

$$\xi^k \exp\left(-t\lambda|\xi|^{2\alpha}\right) \in L^\infty(\mathbb{R}) \ .$$

Let $k \in \mathbb{N}$ be an arbitraty derivative order. The Fourier transform of the $k$th weak derivative of our filtered image

$$\mathcal{F} D^{(k)} \left(T_t f\right) = i^k \xi^k \exp\left(-t\lambda|\xi|^{2\alpha}\right) \hat{f} \qquad (17)$$

is in $L^2(\mathbb{R})$ as the product of $\hat{f}$ with a bounded function. We have shown that $T_t f \in H^k(\mathbb{R})$. □

Since $k$ was arbitrary in the last proposition we know with the Sobolev embedding theorem (see [12–Chapter 4, Proposition 1.3]) that for each $m \in \mathbb{N}$ there is an $u \in C^m(\mathbb{R})$ with $u = T_t f$ almost everywhere. In that sense one could say that the results of such filtering processes are arbitrary smooth for all stopping times $t > 0$. Furthermore, linear diffusion filtering fulfills a choice of scale-space properties.

**Proposition 4.4. (Scale-Space Properties of Diffusion)**

1. *The set of linear diffusion operators* $\{\mathcal{T}_t : t \geq 0\}$ *is a semigroup. We have* $\mathcal{T}_0 = \mathcal{I}$ *and* $\mathcal{T}_{t_1}\mathcal{T}_{t_2} = \mathcal{T}_{t_1+t_2}$ *for all* $t_1, t_2 \geq 0$.
2. *For all* $t \geq 0$ *the average grey value is invariant under* $\mathcal{T}_t$.
3. *The continuous filtering operator is translational invariant.*

*Proof.* 1. Since $G_\lambda^\alpha(\cdot, 0) = \exp(0) = 1$ it is clear that $\mathcal{T}_0 = \mathcal{I}$. For $t_1, t_2 > 0$ and $\xi \in \mathbb{R}$ one can directly verify $G_\lambda^\alpha(\xi, t_1) G_\lambda^\alpha(\xi, t_2) = G_\lambda^\alpha(\xi, t_1 + t_2)$. In the case of a single order $\alpha$ we have with the second statement of Lemma 2.1

$$\begin{aligned}\mathcal{T}_{t_1}\mathcal{T}_{t_2} &= \mathcal{F}^{-1}\mathcal{M}_{G_\lambda^\alpha(\cdot,t_1)}\mathcal{F}\mathcal{F}^{-1}\mathcal{M}_{G_\lambda^\alpha(\cdot,t_2)}\mathcal{F}\\ &= \mathcal{F}^{-1}\mathcal{M}_{G_\lambda^\alpha(\cdot,t_1+t_2)}\mathcal{F}\\ &= \mathcal{T}_{t_1+t_2} \quad .\end{aligned}$$

The same proof also works for multiple filter orders.

2. Average grey value invariance is guaranteed by $G_\lambda^\alpha(0, t) = 1$ for all $t, \alpha, \lambda > 0$.
3. Translational invariance follows directly from the representation of the operator $\mathcal{T}_t$ as convolution with $p$ as in (15). □

Scale invariance is not given in the framework considered above: To achieve this property we have to restrict ourselves to a single derivative order.

**Proposition 4.5. (Scale Invariance of Diffusion)** *With only a single derivative order, the diffusion filter* $\mathcal{T}_f := \mathcal{F}^{-1}\mathcal{M}_{G_\lambda^\alpha(\cdot,t)}\mathcal{F}$ *is scale invariant in the following sense: For every* $\sigma > 0$ *and every* $t > 0$ *there is a* $\tilde{t} > 0$ *such that*

$$\left(\mathcal{T}_t f\left(\frac{\cdot}{\sigma}\right)\right)(x) = \left(\mathcal{T}_{\tilde{t}} f(\cdot)\right)\left(\frac{x}{\sigma}\right) \quad .$$

*Proof.* It can be shown by elementary calculations that $\tilde{t} = \frac{t}{\sigma^{2\alpha}}$ is the unique value satisfying the above condition. Since $\tilde{t}$ depends on the order $\alpha$ such a time can not exist for a combination of different orders. □

## 5 Semi-discrete Linear Filtering

For practical purposes a space-discrete formulation of generalised linear filtering can be very useful. In this section we give a matrix representation for the filters which can be understood as a finite-dimensional analogue of the operators given above. In correspondence to the operator $\mathcal{F}$ we define the discrete Fourier transform $F \in \mathbb{C}^{n\times n}$ as the matrix

$$F := \frac{1}{\sqrt{n}} \left( \exp\left( -\frac{2\pi i \left(j - \frac{n}{2}\right) k}{n} \right) \right)_{j,k=0,\dots,n-1} . \tag{18}$$

Since the rows of $F$ are orthonormal in $\mathbb{C}^n$, $F$ is unitary and its inverse is given by its complex conjugated and transposed matrix $\overline{F}^T$. The matrix-vector product of $F$ with $g \in \mathbb{R}^n$ yields the Fourier coefficients $Fg =: \left(\hat{g}_{-n/2}, \dots, \hat{g}_{n/2-1}\right)^T \in \mathbb{C}^n$. We define the analogue to the multiplication operator $\mathcal{M}$ as the diagonal matrix

$$M_f := \operatorname{diag}\left( f\left( \frac{2\pi \left(j - \frac{n}{2}\right)}{n} \right) \right)_{j=0,\dots,n-1} \tag{19}$$

which multiplies a vector with the values of a function $f : [-\pi, \pi) \longrightarrow \mathbb{C}$ at the equidistant grid points in the Fourier domain.

**Definition 5.1. (Semi-discrete Regularisation and Diffusion Matrices)** *As space-discrete analogues to (9) and (16), for $\alpha, \beta, \lambda \in \mathbb{R}^m_+$ and $t > 0$ we define the regularisation matrix $R^\alpha_\beta := \overline{F}^T M_{r^\alpha_\beta} F$ and the linear diffusion matrix via $T_t := \overline{F}^T M_{G^{\alpha_m}_{\lambda_m}(\cdot,t)} \cdot \ldots \cdot M_{G^{\alpha_1}_{\lambda_1}(\cdot,t)} F$.*

In the semi-discrete case the scale-space properties slightly differ from the continuous ones considered in the last sections. Since the discretisation in space leads to a band-limiting we observe not only average grey value invariance but also convergence towards a constant signal.

**Proposition 5.2. (Scale-Space Properties of Regularisation)**

1. *Semi-discrete regularisation is linear.*
2. *The average grey value is invariant under the operators $R^\alpha_\beta$ for all $t \geq 0$. For $\beta \to \infty$ in all components the solution converges towards the average grey value, i. e. $\lim_{\beta\to\infty} R^\alpha_\beta f = (\mu, \dots, \mu)^T$ with $\mu := \frac{1}{n} \sum_{k=1}^n f_k$ .*

*Proof.* The average grey value can be written as $\hat{f}_0 = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} f_k$. This coefficient is left unchanged by the diagonal matrices $M_{r^\alpha_\beta}$ since $r^\alpha_\beta(0) = 1$. Thus claimed convergence follows from $\lim_{\beta\to\infty} r^\alpha_\beta(\xi) = 0$ for all. □

In addition to these properties the diffusion operators form a semigroup.

**Proposition 5.3. (Scale-Space Properties of Diffusion)**

1. *Semi-discrete diffusion is linear.*
2. *The set of operators $\{T_t : t \geq 0\}$ is a semigroup.*
3. *The average grey value is invariant under the operators $T_t$ for all $t \geq 0$, and we have convergence towards the average grey value for $t \longrightarrow \infty$.*

*Proof.* The proof of the second statement is analogous to the proof of Prop. 4.4 exchanging the operators $\mathcal{F}$ and $\mathcal{M}$ by their finite-dimensional counterparts $F$ and $M$. The third statement is proven as in the regularisation case. □

## 6 Numerical Examples

In the first numerical experiment we investigate the possibility of building linear combinations with different derivative orders such that the regularisation and diffusion filters satisfy a maximum-minimum property. Knowing from Section 4 that combinations of two orders are no longer scale-invariant we try to preserve one scale-space property at the expense of the other. To reduce the number of possible combinations we consider diffusion equations of the form

$$\frac{\partial}{\partial t}u = -\left(\sqrt{-\frac{\partial^2}{\partial x^2}} + \beta\left(-\frac{\partial^2}{\partial x^2}\right)^{\alpha}\right)u \tag{20}$$

and the corresponding regularisation. For $\alpha$ between 1.5 and 8, we started with $\beta = 0$ and increased it as long as nonnegative convolution kernels were obtained. The maximal values of $\beta$ are shown in Fig. 3. This experiment shows the usefulness of the Poisson scale-space: Using a Gaussian scale-space instead makes it impossible to find a weight $\beta \neq 0$ that leads to a nonnegative combination. In that sense the fractional order scale-space has a clear advantage in comparison with the integer order ones.

In our second experiment we study the variance diminishing properties of different filters $\mathcal{R}$ and $\mathcal{T}$. Fig. 4 shows the variance of a white noise signal depending on regularisation weight / stopping time. We visualise the parameters needed for reducing the variance of a white noise and a step edge signal to half of its value in Fig. 5. The experiments show a similar behaviour of regularisation and corresponding diffusion techniques in terms of variance reduction. We note that higher orders lead to the same variance decay with smaller stopping times.

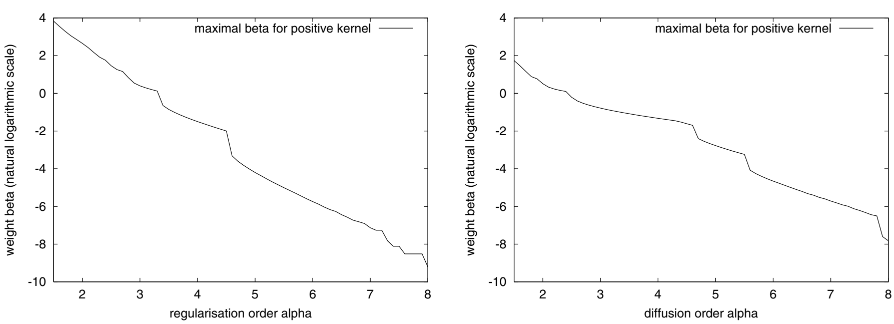


**Fig. 3.** Positive combinations of derivatives of order 0.5 with higher orders. Left: Regularisation. Right: Diffusion filtering

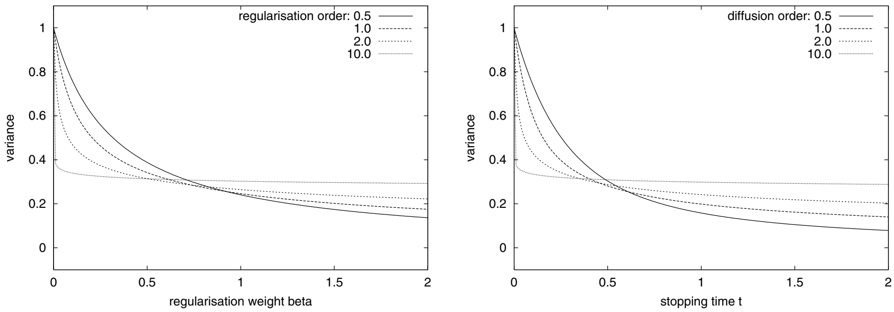


**Fig. 4.** Variance diminishing properties of fractional order regularisation and diffusion filtering. Variance depending on regularisaton weight/diffusion stopping time. Left: Regularisation. Right: Diffusion filtering

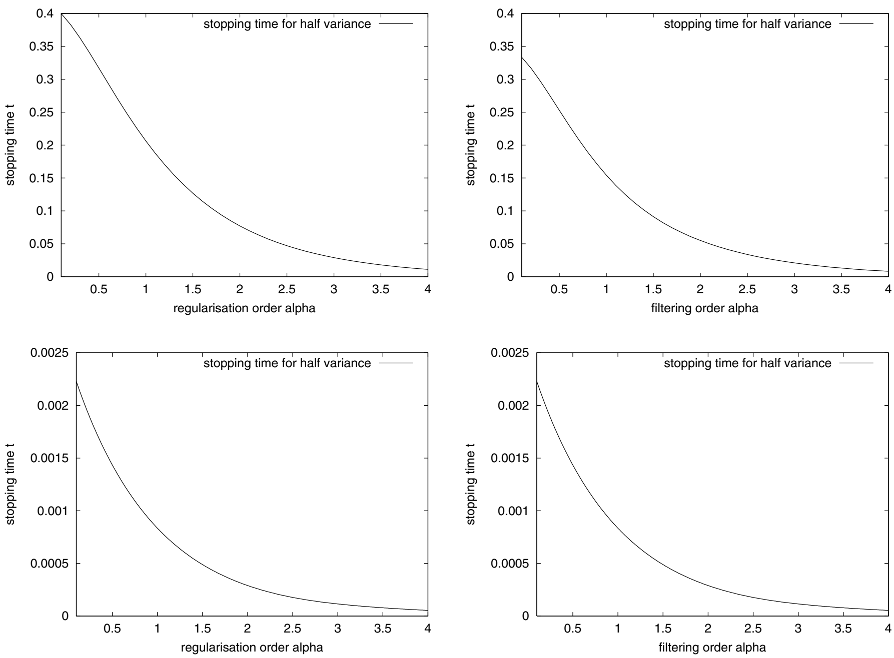


**Fig. 5.** Regularisation weight/diffusion stopping time for reducing the variance to half its value. Left column: Regularisation. Right column: Diffusion filtering. Top row: Experiment for white noise signal. Bottom row: Experiment for step edge signal

## 7 Conclusion

In this paper we have discussed regularisation techniques and diffusion methods that involve sums of fractional derivative orders. With respect to scale-space properties, fractional diffusion satisfies Iijima's axioms of linearity, translation invariance and semigroup property. If a single fractional order is used, scale invariance is satisfied as well. We have shown that both fractional diffusion and fractional regularisation are $L^2$-stable in the sense that the norms of the corresponding operators are bounded by 1. With respect to regularity, the regularisation approaches gain twice the minimal derivative order, while the fractional diffusion admits arbitrarily smooth solutions. For the first time in the context of $\alpha$-scale-spaces, we have also presented a space-discrete theory that is in formal analogy to the continuous framework. Moreover, it gives convergence towards the average grey value, if the diffusion time / regularisation parameter tends to infinity. To our knowledge, all papers on $\alpha$-scale-spaces focus their attention to the case $0 < \alpha \leq 1$, since this guarantees nonnegativity and a maximum-minimum principle. However, we have shown that it is possible to construct combinations of Poisson scale-space and diffusion scale-spaces of order $\alpha > 1$ that satisfy this principle as well. With Gaussian scale-space instead of Poisson scale-space, this is not possible. Similar statements also hold for the corresponding regularisation processes. From a practical viewpoint, we have studied the decay rates of the variance as a function of the fractional order. These studies have shown that higher orders reveal higher variance diminishing properties. In our ongoing and future work we intend to find out which of the scale-space and regularity properties of the linear methods of this paper can be generalised to nonlinear processes with higher-order derivatives.

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