Relativistic Scale-Spaces

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Abstract. In this paper we extend the notion of Poisson scale-space. We propose a generalisation inspired by the linear parabolic pseudodifferential operator $\sqrt{-\Delta + m^2} - m$, $0 \leq m$, connected with models of relativistic kinetic energy from quantum mechanics. This leads to a new family of operators $\{Q_t^m \mid 0 \leq m, t\}$ which we call relativistic scale-spaces. They provide us with a continuous transition from the Poisson scale-space $\{P_t \mid t \geq 0\}$ (for $m = 0$) to the identity operator I (for $m \longrightarrow +\infty$). For any fixed $t_0 > 0$ the family $\{Q_{t_0}^m \mid m \geq 0\}$ constitutes a scale-space connecting I and P_{t_0}. In contrast to the α-scale-spaces the integral kernels for Q_t^m can be given in explicit form for any $m, t \geq 0$ enabling us to make precise statements about smoothness and boundary behaviour of the solutions. Numerical experiments on 1D and 2D data demonstrate the potential of the new scale-space setting.

Keywords: Kinetic energy, Poisson scale-space, semigroup, pseudodifferential operator.

1 Introduction

The pioneering work of Taizo Iijima [16] in the late fifties, though unrecognised in the western scientific world for decades, marks the actual beginning of modern scale-space theory. Since then the vivid research on scale-space methodologies has brought forward many valuable techniques in image processing and computer vision, as it is documented in numerous articles and books, see [24, 11, 31, 21, 28, 33] and the literature cited there. The Gaussian scale-space is the prototype of a linear scale-space. Its connection to linear diffusion processes was first pointed out by Iijima [17]. However, the field of non-linear diffusion, instigated by the influential work of Perona and Malik [25] also exhibits scale-space properties. These non-linear theories encompass anisotropic diffusion processes [33, 26], morphological operations [32, 6, 18] as well as the evolution of level curves [2, 23, 27, 19]. Non-linear differential equations are the mathematical language to describe these theories [31, 33, 14, 3, 12, 7].

Nevertheless, the exploration of the axiomatic principles of the various scale-space approaches [4, 33, 11, 22, 24, 34] usually emanates from the assumption of

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linearity, that is to say, the validity of the superposition principle. In this linear setting the Gaussian scale-space basically had played the leading role in a one man show until the Poisson scale-space from potential theory has been made popular in image processing by Felsberg and Sommer [10].

Soon after the so-called α-scale-spaces with $\alpha \in [\frac{1}{2}, 1]$ have been advocated to bridge the gap between those two prominent representatives since they are ruled by the pseudodifferential equations $\partial_t u = (-\Delta)^\alpha u$ with initial condition $u(x,0) = f(x)$, (for more details and a histortic overview consult the very comprehensive article [8] by Duits et. al. and the literature cited therein). In this setting $\alpha = 0$ produces the family of identity operators I, $\alpha = \frac{1}{2}$ corresponds to the Poissonian, while $\alpha = 1$ delivers the Gaussian version of a linear scale-space. For the later two cases explicit integral representation formulas are known utilising the Poisson and the Gaussian kernel.

The primary tool for the investigation of the α-scale-spaces are Fourier methods since, unfortunately, no explicit integral kernel can be determined. In our paper, however, we propose a counterpart to α-scale-spaces that admits explicit kernel representations. We generalise the Poisson scale-space to a novel scale-space by exploiting the properties of a pseudodifferential operator known from Schrödinger operators in relativistic quantum mechanics [20]. The pseudodifferential operators in question read

$$\sqrt{-\Delta + m^2} - m,$$

and represent the kinetic energy operators in relativistic systems with $m > 0$ denoting mass. Therefore we will refer to these novel scale-spaces as *relativistic scale-spaces* in the sequel. Though heavily taking advantage of spectral methods during the theoretical investigation of this family of operators (indexed by m) we emphasise that the associated integral kernels can be computed explicitly. The knowledge of these kernels enables us to employ techniques from analysis to prove regularity and a maximum-minimum-principle for the solutions of the associated evolution equation.

In the sequel $\mathcal{F}(f)$ will denote the Fourier transform of a function $f \in L^2(\mathbb{R}^n)$ given by

$$\mathcal{F}(f)(k) = \int_{\mathbb{R}^n} e^{-2\pi i k \cdot x} f(x)\, dx\,.$$

The structure of our paper is as follows: After a very brief motivating account of some basic facts about Poisson and Gaussian scale-space we introduce and study the relativistic scale-spaces. Section 3 reports on experiments displaying the potential and limitations of the novel scale-spaces while a summary and an outlook for future research in Section 4 conclude the paper.

2 Relativistic Scale-Spaces

We recall [9, 20] that the action of the Laplace operator $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ on functions in the Fourier domain is multiplication by $-4\pi^2 |k|^2$, i.e.

$$\mathcal{F}(\Delta f) = -4\pi^2|k|^2\mathcal{F}(f)$$

while the convolution with the heat or Gaussian kernel $G(x,t,y)$ means multiplication with $e^{-t\ 4\pi^2|k|^2}$, $\mathcal{F}(G * f) = e^{-t\ 4\pi^2|k|^2}\mathcal{F}(f)$ providing solutions of the heat equation $\partial_t u = \Delta u$.

Furthermore, the action of the pseudodifferential operator $\sqrt{-\Delta}$ is multiplication by $-2\pi|k|$, while convolution with the Poisson kernel $P(\cdot,t)$ means multiplication with $e^{-t\ 2\pi|k|}$ in the Fourier domain. The Poisson kernel appears as the inverse Fourier transform $\mathcal{F}^{-1}$ of $e^{-t\ 2\pi|k|}$:

$$P(x-y,t) = \mathcal{F}^{-1}\big(e^{-t\ 2\pi|\cdot|}\big) = \int_{\mathbb{R}^n} e^{-t\,2\pi|k|+2\pi ik\cdot(x-y)}\,dk\,.$$

This integral can be evaluated in every dimension n yielding the well-known explicit formula for the Poisson kernel [29]

$$P(x-y,t) = \Gamma\left(\frac{n+1}{2}\right)\frac{1}{\pi^{\frac{n+1}{2}}}\frac{t}{(t^2+|x-y|^2)^{\frac{n+1}{2}}}\,. \tag{1}$$

The kernel itself and all convolutions $P(\cdot,t) * f$ with suitable functions f solve in a certain sense the pseudodifferential equation $\partial_t u = \sqrt{-\Delta} u$. The heat and the Poisson kernel generate the Gaussian, resp., the Poisson scale-space.

This can be generalised as follows: In quantum mechanics the pseudodifferential operator $L := \sqrt{-\Delta + m^2} - m$ describes the relativistic kinetic energy of a particle with mass $m \geq 0$ [20] seemingly extending the Poisson operator. In Fourier space this operator acts on function by multiplication with $\sqrt{|2\pi k|^2 + m^2} - m$ as a straightforward computation shows. According to standard spectral methods the corresponding integral operator in Fourier space reads

$$e^{-t\ (\sqrt{|2\pi k|^2+m^2}-m)}\,.$$

The inverse Fourier transform of this exponential

$$T_m(x-y,t) := \mathcal{F}^{-1}\left(e^{-t\ (\sqrt{4\pi^2|\cdot|^2+m^2}-m)}\right)(x,y)$$

can be calculated explicitly yielding the expression

$$T_m(x-y,t) := 2\left(\frac{m}{2\pi}\right)^{\frac{n+1}{2}} e^{tm}\frac{t}{\left(t^2+|x-y|^2\right)^{\frac{n+1}{4}}} K_{\frac{n+1}{2}}(m\sqrt{t^2+|x-y|^2}) \tag{2}$$

for $(x-y,t) \in R^n \times]0,+\infty[$. Here K_ν stands for the modified Bessel function of the third kind [1, 13]. We briefly sketch the computational steps by pointing out the formulas

$$\int_{S^{n-1}} e^{i\langle\omega,x\rangle}\,d\omega = (2\pi)^{\frac{n}{2}}|x|^{1-\frac{n}{2}}J_{\frac{n}{2}-1}(|x|)$$

and

$$\int_{[0,+\infty[} x^{\nu+1}J_\nu(xs)\,e^{-\alpha\sqrt{x^2+\beta^2}}dx = \sqrt{\frac{2}{\pi}}\alpha\beta^{\nu+\frac{3}{2}}(s^2+\alpha^2)^{-\frac{\alpha}{2}-\frac{3}{4}}s^\nu K_{\nu+\frac{3}{2}}(\beta\sqrt{s^2+\alpha^2})\,,$$

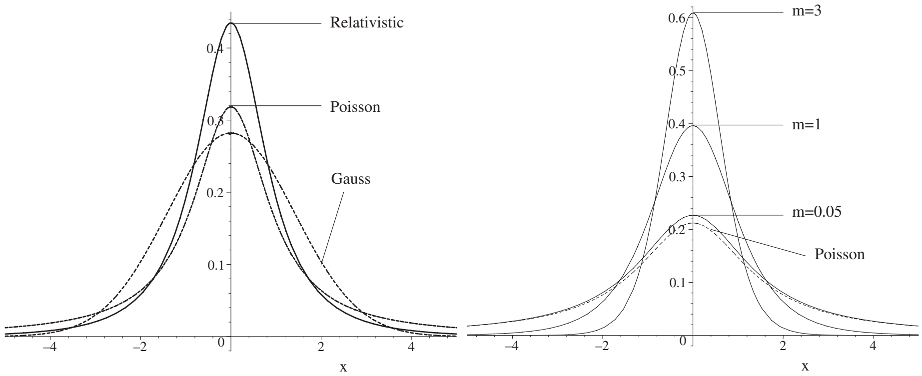

Fig. 1. *Left:* Comparison between different kernels including Poisson, eqn. (1) and relativistic kernel, eqn. (2) in 1D for $y = 0$ and $t = 1$. *Right:* Examples of the relativistic kernel (2) with $m = 3,\ 1,\ ,0.05$ in comparison with the Poisson kernel (1) for $y = 0$ and $t = 1.5$

where J_ν denotes the ν-th order Bessel function. For later use we define the operator Q_t^m on $L^2(\mathbb{R}^n)$ via the convolution

$$Q_t^m f(x) := T_m(\cdot,t) * f\ (x) = \int_{\mathbb{R}^n} T_m(x-y,t) f(y)\, dy\,. \tag{3}$$

2.1 Comparison with the Poisson Kernel

For $m \downarrow 0$ we regain the Poisson kernel which follows from

$$\mathcal{F}(Q_t^m)(k) = e^{-t\ (\sqrt{|2\pi k|^2+m^2}-m)} \longrightarrow e^{-t\ 2\pi|k|} \quad \text{if } m \downarrow 0 \tag{4}$$

for any complex number k together with the continuity of the (inverse) Fourier transform (according to a theorem of P. Levy) [5]. Furthermore, since

$$\mathcal{F}(Q_t^m)(k) = e^{-t\ (\sqrt{|2\pi k|^2+m^2}-m)} \longrightarrow 1 \quad \text{if } m \to +\infty\,,$$

a similar reasoning proves that Q_t^m approximates therefor the identity operator I if m is large. Remarkably, despite the approximation property (4), we learn from the theory of Bessel functions [1, 13] that $K_\nu(x)$ for any $\nu \geq 0$, and hence T_m as a function of x (or of y) decreases exponentially to 0 for x tending to infinity, $|x| \to +\infty$. Figure 1 displays the relativistic kernel for various values of m and also its comparison with a Poisson and a Gaussian kernel.

The relation between Poisson scale-space and the relativistic scale-spaces is sketched in the diagram to the right. $\{Q_t^m \mid 0 \leq t, m\}$ is positioned between The Poisson scale-space $\{P_t \mid 0 \leq t\}$ and $\{I \mid 0 \leq t\}$ including them as limiting cases.

$$\begin{array}{cccc}
+\infty & I_0 & \longrightarrow & I_0 \\
\uparrow & \uparrow & & \uparrow \\
m & Q_0^m = I_0 & \longrightarrow & Q_t^m \\
\uparrow & \uparrow & & \uparrow \\
0 & P_0 = I_0 & \longrightarrow & P_t \\
 & & & \\
 & 0 & \longrightarrow & t
\end{array}$$

2.2 Further Properties of the Relativistic Scale-Spaces

From the theory of contraction semigroups [15] we learn that the operator Q_t^m determines a contraction semigroup on $L^2(\mathbb{R}^n)$. Indeed, in view of Plancherel's theorem, it is enough to verify that the Fourier transforms $\mathcal{F}(Q_t^m) = e^{-t(\sqrt{|2\pi k|^2+m^2}-m)}$ of the family $\{Q_t^m\}$ satisfy the conditions

1. $\mathcal{F}(Q_{s+t}^m)\mathcal{F}(f) = \mathcal{F}(Q)_s^m \mathcal{F}(Q_t^m)\mathcal{F}(f) = \mathcal{F}(Q_t^m)\mathcal{F}(Q_s^m)\mathcal{F}(f)$ for all $s, t \geq 0$.

2. $\|\mathcal{F}(Q_t^m)\mathcal{F}(f) - \mathcal{F}(Q_s^m)\mathcal{F}(f)\|_2 \longrightarrow 0$ for $t \longrightarrow s$.

3. $\mathcal{F}(Q_0^m) = 1$, expressing the fact that $Q_0^m = I$, the identity .

4. $\|\mathcal{F}(Q)\mathcal{F}(f)\|_2 \leq \|\mathcal{F}(f)\|$, the contraction property.

Due to the properties of the exponentials e^{-ct} with $c > 0$ it is not difficult to check that the operator Q_t^m indeed meets these conditions. The associated *generator* is the pseudodifferential operator $L = \sqrt{-\Delta + m^2} - m$ with the Sobolev space $H^1(\mathbb{R}^n)$ as its domain $D(L)$. Here we followed [30] in the definition of the Sobolev spaces

$$H^s(\mathbb{R}^n) := \left\{ u \in L^2(\mathbb{R}^n) \mid \left(1 + |k|^2\right)^{\frac{s}{2}} \mathcal{F}(u) \in L^2(\mathbb{R}^n) \right\} \tag{5}$$

of all functions in $L^2(\mathbb{R}^n)$ and $s \in \mathbb{R}$.

Next we are going to study in some detail the properties of the function $F_m(x, t)$ defined for $f \in L^2(\mathbb{R}^n)$ by

$$F_m(x, t) := Q_t^m f(x) = \int_{\mathbb{R}^n} T_m(x - y, t) f(y)\, dy$$

with $x \in \mathbb{R}^n$ and $t > 0$. Since the Bessel functions $K_\nu(x)$ are analytic for $0 < x$, the following result is not surprising.

Proposition 2.1. *F_m is analytic in $\mathbb{R}^n \times]0, \infty[$ for any function $f \in L^2(\mathbb{R}^n)$.*

Proof: Thanks to the analyticity of K_ν the function T can be expanded locally in a multivariate power series to the effect that the exchange of integration and summation yields a corresponding expansion for F_m.

Having the explicit integral kernel at our disposal will enable us to study the boundary behaviour of $F_m(x, t)$ as $t \downarrow 0$. To this end we need the next lemma.

Lemma 2.2. *For any $z \geq 0$ and $\nu \geq -\frac{1}{2}$ the following estimate holds:*

$$K_\nu(z) \leq \frac{\Gamma(\nu)}{2} \left(\frac{2}{z}\right)^\nu . \tag{6}$$

Proof: Taking advantage of an integral representation in [13], page 958, and using the well-known definition of the Γ-function we obtain

$$\begin{aligned} K_\nu(z) &= \sqrt{\pi}\left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+\frac{1}{2})}\int_1^\infty e^{-tz}(t^2-1)^{\nu-\frac{1}{2}}\,dt \\ &\le \sqrt{\pi}\left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+\frac{1}{2})}\int_0^\infty e^{-tz}t^{2\nu-1}\,dt \\ &= \sqrt{\pi}\left(\frac{z}{2}\right)^\nu \frac{1}{\Gamma(\nu+\frac{1}{2})}\cdot\frac{2}{\sqrt{\pi}}\left(\frac{2}{z}\right)^{2\nu}\Gamma(\nu)\Gamma\left(\nu+\frac{1}{2}\right) = \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu . \end{aligned}$$

This inequality is asymptotically ($\sim$) sharp since for $z \longrightarrow 0$,[1], page 375: $K_\nu(z) \sim \frac{\Gamma(\nu)}{2}\left(\frac{2}{z}\right)^\nu$. With this at our disposal we can proceed to the result stating that $F_m(\cdot,t) = Q_t^m\, f$ has exactly the same boundary behaviour as the corresponding functions stemming from the Gaussian or Poisson scale-space.

Theorem 2.3. *Suppose that f is a continuous and bounded on $\mathbb{R}^n$, $f \in \mathcal{C}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then the function $F_m(x,t) = Q_t^m f(x)$ satisfies the pseudodifferential equation*

$$\partial_t F_m = (\sqrt{-\Delta + m^2} - m)F_m \tag{7}$$

for any $t > 0$ with the initial condition $\lim_{t\downarrow 0} F_m(\cdot,t) = f$.

Proof: That $F_m(x,t)$ satisfies (7) follows from the analysis above remembering that the Fourier transform of $\sqrt{-\Delta+m^2} - m$ is given by $\sqrt{4\pi^2|k|^2+m^2} - m$. Also, as stated above, the corresponding solution operator is given by Q_t^m. In order to prove the claimed boundary behaviour we observe that

$$\int_{\mathbb{R}^n} T_m(x-y,t)\,dy = e^{-t(\sqrt{0+m^2}-m)} = 1 \tag{8}$$

for all $x \in \mathbb{R}^n$ and $t > 0$, since the integral at the left side can be considered as the Fourier transform $\mathcal{F}(T_m(\cdot,t))$ of $T_m(\cdot,t)$ evaluated at $k = 0$.

Next we fix a $x_0 \in \mathbb{R}^n, \epsilon > 0$, and choose $\delta > 0$ so small that if

$$|y - x_0| < \delta \text{ for } y \in \mathbb{R}^n \text{ then } |f(y) - f(x_0)| < \epsilon\,. \tag{9}$$

For $(x,t) \in \mathbb{R}^n \times]0,+\infty[$ with $|(x,t)-(x_0,0)| < \frac{\delta}{2}$ we obtain the estimate

$$\begin{aligned} |F_t(x,t) - f(x_0)| &= \left|\int_{\mathbb{R}^n} T_m(x-y,t)f(y)\,dy - f(x_0)\cdot\int_{\mathbb{R}^n} T_m(x-y,t)\,dy\right| \\ &\le \int_{B(x_0,\delta)} T_m(x-y,t)|f(y)-f(x_0)|\,dy \\ &\quad + \int_{\mathbb{R}^n\setminus B(x_0,\delta)} T_m(x-y,t)|f(y)-f(x_0)|\,dy \\ &=: I_1 + I_2\,. \end{aligned}$$

The equality (8) and the restriction on y in (9) yield

$$I_1 \leq \int_{\mathbb{R}^n} T_m(x-y,t)\varepsilon\, dy = \varepsilon\,.$$

If additionally $|y-x_0| \geq \delta$ we find

$$\begin{aligned}|y-x_0| &\leq |(y,0)-(x,t)| + |(x,t)-(x_0,0)| \\ &\leq |(y,0)-(x,t)| + \frac{\delta}{2} \leq |(y,0)-(x,t)| + \frac{1}{2}|y-x_0|\end{aligned}$$

which yields $|(y,0)-(x,t)| \geq \frac{1}{2}|y-x_0|$. This gives way to the estimates

$$\begin{aligned}I_2 &\leq 2\|f\|_\infty \int_{\mathbb{R}^n\setminus B(x_0,\delta)} T_m(x-y,t)\, dy \\ &= 4\|f\|_\infty \left(\frac{m}{2\pi}\right)^{\frac{n+1}{2}} e^{-tm} \int_{\mathbb{R}^n\setminus B(x_0,\delta)} \frac{t}{|(x,t)-(y,0)|^{\frac{n+1}{2}}} K_{\frac{n+1}{2}}(m|(x,t)-(y,0)|)\, dy \\ &\leq 4\|f\|_\infty \left(\frac{m}{2\pi}\right)^{\frac{n+1}{2}} e^{-tm} \int_{\mathbb{R}^n\setminus B(x_0,\delta)} \frac{t}{\left(\frac{1}{2}|x_0-y|\right)^{\frac{n+1}{2}}} K_{\frac{n+1}{2}}\left(\frac{m}{2}|x_0-y|\right) dy \\ &\leq 2\|f\|_\infty \left(\frac{m}{\pi}\right)^{\frac{n+1}{2}} \Gamma(\nu)\, e^{-tm}\cdot t \int_{\mathbb{R}^n\setminus B(x_0,\delta)} \frac{1}{|x_0-y|^{n+1}}\, dy \longrightarrow 0,\ \text{as } t \downarrow 0\,.\end{aligned}$$

The second inequality follows from the fact that $\frac{1}{|\cdot|}^{\frac{n+1}{2}}$ and K_ν are decreasing functions on $]0,\infty[$ while the last inequality is due to estimate (6) in lemma (2.2). Hence, we deduce $|F_m(x,t) - f(x_0)| \leq I_1 + I_2 \leq 2\varepsilon$ as soon as $|(x,t)-(x_0,0)|$ is sufficiently small proving the continuity of F_m on the closed set $\mathbb{R}^n \times [0,+\infty[$. Summarising the analysis above we state

Proposition 2.4. *1. The families of operators* $\{Q_t^m \mid t \geq 0\}$ *form for any fixed* $m \geq 0$ additive semigroups.

2. For every $t \geq 0$ *the average grey-value is preserved under the action of* Q_t^m.

3. The operators Q_t^m *are translational invariant.*

For large values of m the relativistic scale-spaces apparently approximate the trivial scale-space $\{I_t \mid t \geq 0\}$ with $I_t = I_0$ for all $t > 0$, while for small m they are very close to the Poisson scale-space.

However, with a fixed t_0 the family $\{Q_{t_0}^m \mid m \geq 0\}$ is also a scale-space, but it has no longer an additive semigroup property: $Q_t^{m_1} Q_t^{m_2} \neq Q_t^{m_1+m_2}$.

We mention briefly that $\{Q_t^{f(t)} \mid t \geq 0\}$ with an arbitrary decreasing function $f : [0,+\infty[\longrightarrow [0,+\infty[$ also describes a scale-space relying on a non-additive semigroup.

3 Numerical Experiments with Relativistic Scale-Spaces

In this section we present some numerical experiments to visualise the properties of relativistic scale-spaces. We have implemented the methods in the Fourier domain using the Discrete Fourier Transform (DFT) or Fast Fourier Transform (FFT) for suitable data dimensions. The filtering operation then can be performed as a multiplication of the Fourier coefficients with $\mathcal{F}(Q_t^m)$. Figures 2 and 3 show the simplifying effect of the relativistic scale-space in 1D and 2D for fixed stopping time t but varying parameter m. Vice versa, Fig. 4 shows a time evolution for fixed value of m and increasing time t. For $m = 0$ this we would obtain the Poisson scale-space.

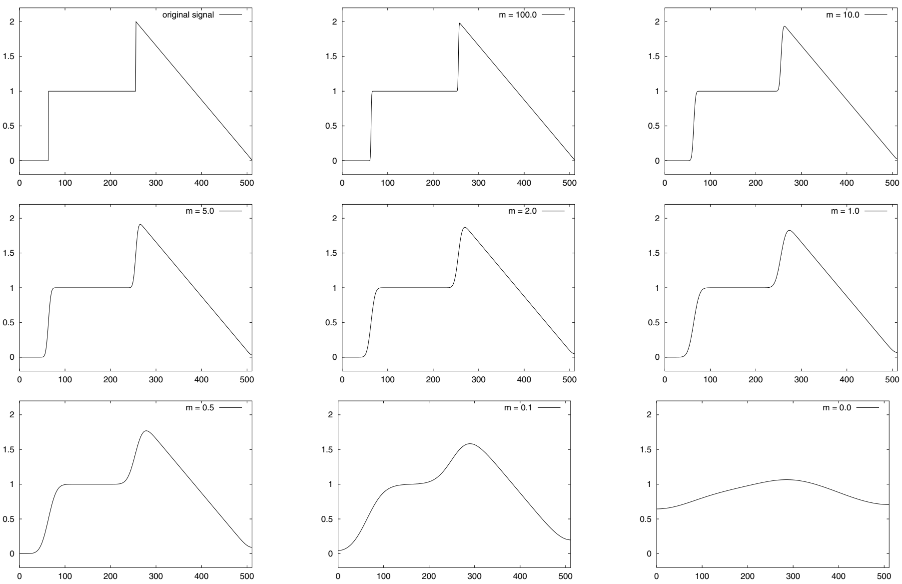

Fig. 2. Relativistic scale-space in 1D. The stopping time $t = 100$ is fixed. *Top left:* Initial signal. The mass m decreases from the top middle to the bottom right

4 Conclusion

The goal of this paper is to propose the novel two-parameter family of relativistic scale-spaces as a generalisation of the well-known Poisson scale-space, and as a counterpart to the α-scale-spaces. As such the relativistic scale-spaces are generated by pseudodifferential operators and they provide a continuous interpolation between the identity operator and the Poisson scale-space. Unlike the α-scale-spaces these new scale-spaces admit integral representations with

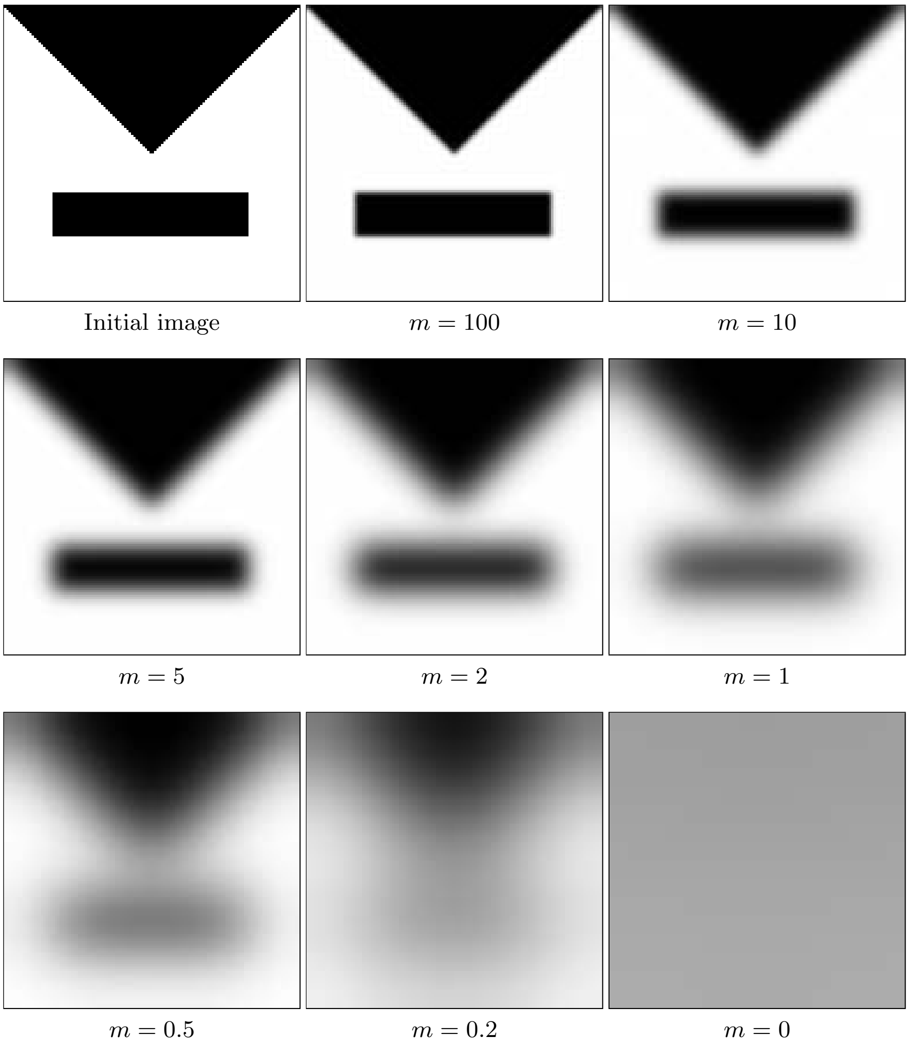

Fig. 3. Relativistic scale-space in 2D. The stopping time $t = 100$ is fixed. *Top left:* Initial image. The mass m decreases from top middle to bottom right

explicit convolution kernels involving Bessel functions. This paves the way to prove analyticity and continuous extendability of the solutions of the relativistic pseudodifferential equations.

This work evidences once more that spectral methods for pseudodifferential operators are very useful for the study and extension of scale-space concepts. Further generalisations of the relativistic scale-spaces in the framework of pseudodifferential operators are close at hand. For instance, the “α-variant“ generated by $(-\Delta + m^{\frac{1}{\alpha}})^{\alpha} - m$ is the subject of ongoing research. Future research will also encompass the search for variational formulations hoping to discover

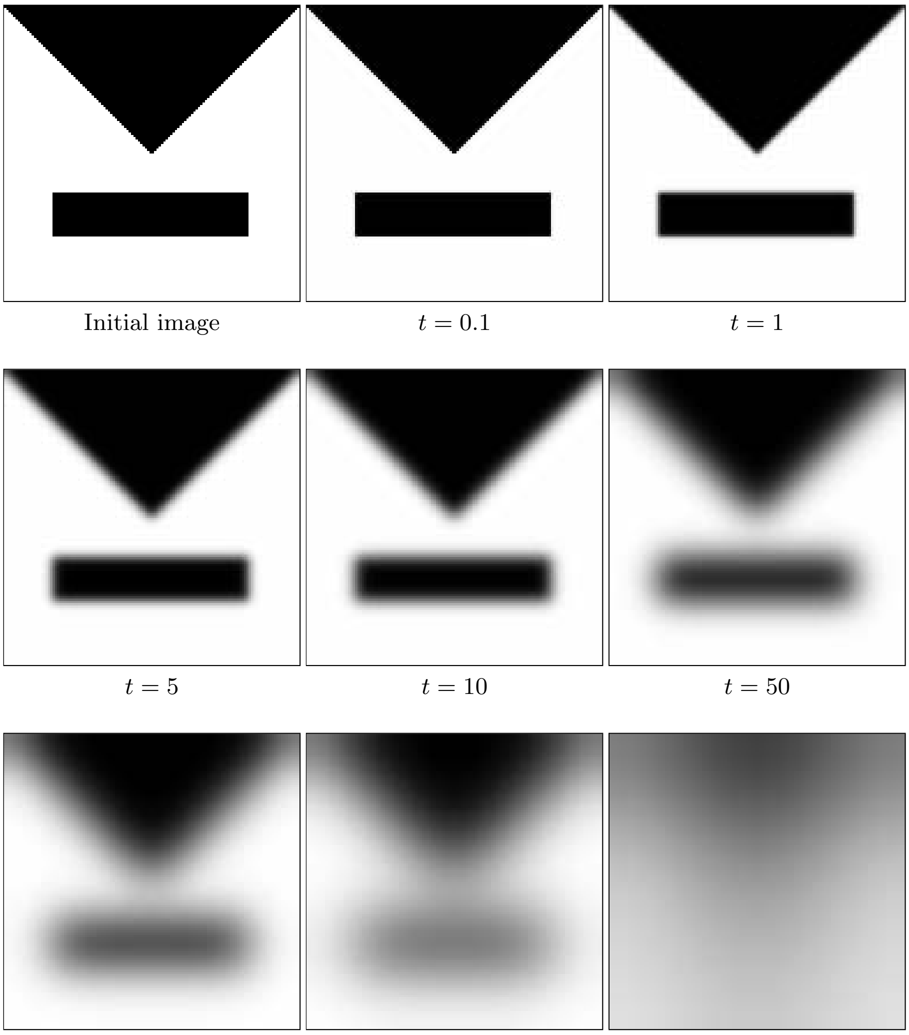

Fig. 4. Relativistic scale-space in 2D. The parameter $m = 1.0$ is fixed, and the time increases from top left to bottom right

new valuable tools for image filtering, and to enhance insight into the structure of scale-spaces.

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