An Explanation for the Logarithmic Connection between Linear and Morphological Systems

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Abstract. Since the introduction of the slope transform by Dorst/van den Boomgaard and Maragos as the morphological equivalent of the Fourier transform, people have been surprised about the almost logarithmic relation between linear and morphological system theory.
This article gives an explanation by revealing that morphology in essence is linear system theory in a specific algebra. While classical linear system theory uses the standard $(+, \times)$-algebra, the morphological system theory is based on the idempotent $(\max, +)$-algebra and the $(\min, +)$-algebra. We identify the nonlinear operations of erosion and dilation as linear convolutions $*_e$ and $*_d$ induced by these idempotent algebras. The slope transform in the $(\max, +)$-algebra, however, corresponds to the logarithmic multivariate Laplace transform in the $(+, \times)$-algebra. We study relevant properties of this transform and its links to convex analysis. This leads to the definition of the so-called Cramer transform as the Legendre-Fenchel transform of the logarithmic Laplace transform. Originally known from the theory of large deviations in stochastics, the Cramer transform maps standard convolution to $*_e$-convolution, and it maps Gaussians to quadratic functions.
The article is a step towards the unification of linear and morphological system theories on the basis of a general linear system theory in an appropriate algebra.

Keywords: linear system theory, morphology, convex analysis, MAXPLUS algebra, MINPLUS algebra, slope transform, Cramer transform.

1 Introduction

Linear system theory is a successful and well established field in signal and image processing [6, 14, 15, 29]. In the n-dimensional case, shift invariant linear filters can be described as convolutions of some signal $f : \mathbb{R}^n \to \mathbb{R}$ with a kernel function $b : \mathbb{R}^n \to \mathbb{R}$:

$$f * b(x) := \int_{\mathbb{R}^n} f(x-y)\, b(y)\, dy.$$

By means of the Fourier transform

$$\hat{f}(u) := \mathcal{F}[f](u) := \int_{\mathbb{R}^n} f(x)\, e^{-i2\pi u^\top x}\, dx$$

and its backtransformation

$$\mathcal{F}^{-1}[g](x) := \int_{\mathbb{R}^n} g(u)\, e^{i2\pi u^\top x}\, du$$

one may conveniently compute a convolution in the spatial domain via a simple product in the Fourier domain:

$$\mathcal{F}[f * b] \;=\; \mathcal{F}[f] \,\cdot\, \mathcal{F}[b]\,.$$

In this context, Gaussians

$$K_\sigma(x) \;:=\; \frac{1}{(2\pi\sigma^2)^{n/2}}\, e^{-\frac{x^\top x}{2\sigma^2}}$$

play an important role as convolution kernels: They are the only separable and rotationally invariant function that preserve their shape under the Fourier transform. Convolutions of a signal f with the family $\{K_\sigma \mid \sigma > 0\}$ of Gaussians create the Gaussian scale-space [20, 37, 38], a multiscale representation that is useful in pattern recognition, image processing and computer vision [11, 21, 24, 35]. Figure 1(a) shows an example.

Mathematical morphology is an interesting nonlinear alternative to linear systems theory [16, 27, 32–34]. It has been applied successfully to a large number of fields including cell biology, computer-aided quality control, mineralogy, remote sensing and medical imaging. Morphology is based on two fundamental processes: dilation and erosion. In the case of nonflat morphology, the dilation resp. erosion of some function $f : \mathbb{R}^n \to \mathbb{R}$ with a structuring function $b : \mathbb{R}^n \to \mathbb{R}$ can be defined as follows (see e.g. [25, 36]):

$$\begin{aligned}(f \oplus b)\,(x) &:= \sup\,\{f(y) + b(x-y) \mid y \in \mathbb{R}^n\},\\ (f \ominus b)\,(x) &:= \inf\;\{f(y) - b(y-x) \mid y \in \mathbb{R}^n\}.\end{aligned}$$

Dorst and van den Boomgaard [9] and Maragos [25] developed independently and simultaneously a morphological system theory that closely resembles linear system theory. Following [9], one may generalise the dilation to the tangential dilation via

$$(f \,\check{\oplus}\, b)(x) \;:=\; \operatorname*{stat}_y \Big(f(y) + b(x-y)\Big)$$

with $\operatorname*{stat}_y f(y) := \{f(z) \mid \nabla f(z) = 0\}$. Then the morphological equivalent to the Fourier transform is given by the slope transform

$$\mathcal{S}[f](u) := \operatorname*{stat}_x (f(x) - u^\top x),$$

a transformation that is closely related to the Legendre transform and the Young–Fenchel conjugate in convex analysis. Its backtransformation is given by

$$\mathcal{S}^{-1}[g](x) \;=\; \operatorname*{stat}_{u}(g(u) + u^\top x).$$

The slope transform allows to replace the tangential dilation by simple addition in the slope domain:

$$\mathcal{S}[f \check{\oplus} b] \;=\; \mathcal{S}[f] + \mathcal{S}[b]\,.$$

Paraboloids

$$b(x,t) \;=\; -\frac{x^\top x}{4t} \qquad (t > 0)$$

are those structuring functions in morphological system theory that play a comparable role as Gaussians in linear system theory [36]: They are the only rotationally invariant and separable structuring functions that maintain their shape under the slope transformation. The corresponding dilation and erosion scale-spaces are depicted in Figure 1(b) and (c). For a detailed analysis of their scale-space properties, we refer to Jackway and Deriche [23]. Morphological scale-spaces with paraboloids as structuring functions are useful for computing Euclidean distance transformations [36], for image enhancement [31] and for multiscale segmentation [22].

From these discussions we observe that there seems to be an almost logarithmic connection between linear and morphological system theory. The structural similarities between linear and morphological processes have triggered Florack *et al.* [12, 13] to construct a one-parameter process that incorporates Gaussian scale-space, and both types of morphological scale-spaces as limiting processes. Heijmans and van den Boomgaard [17, 18] have investigated unifying algebraic definitions of scale-space concepts that include a number of linear and morphological approaches (cf. also [2]).
However, in spite of these very interesting contributions, the reason for the almost logarithmic connection between linear and morphological systems has not been discovered so far. To address this problem is the topic of the present paper.

We provide an explanation for the structural analogies between linear and morphological systems by revealing that morphology in essence is linear system theory in a specific algebra. While classical linear system theory uses the standard $(+, \times)$-algebra, the morphological system theory is based on the idempotent $(\max, +)$-algebra and the $(\min, +)$-algebra. This allows us to identify the nonlinear operations of erosion and dilation as linear convolutions $*_e$ and $*_d$ induced by these idempotent algebras. In this sense, morphology may be regarded as linear system theory in disguise.

These algebraic structures have already numerous interesting applications [4]: so-called discrete event dynamic systems (DEDS) can be modeled as *linear* systems with respect to these algebras. Discrete event dynamic systems in this algebraic formulation are used to find shortest paths in networks or to solve

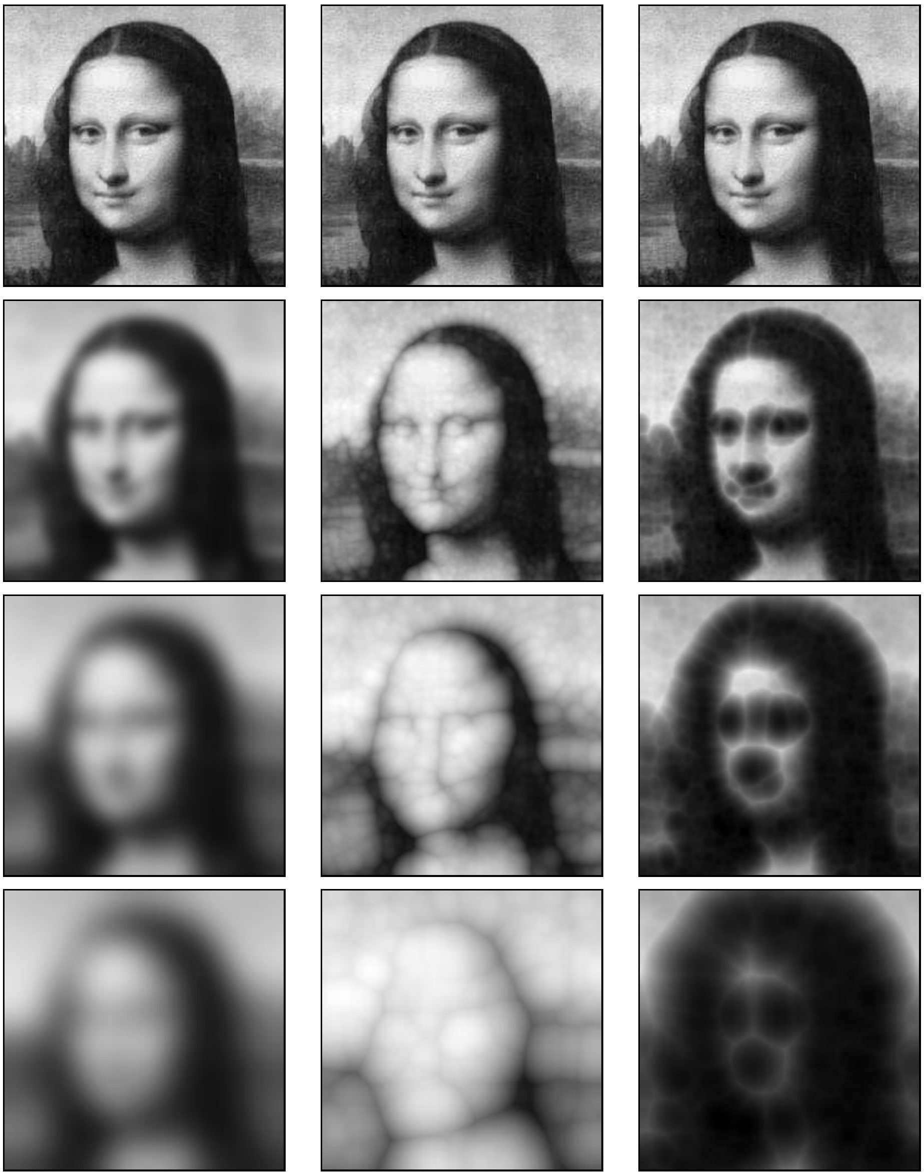

Fig. 1. Linear and morphological scale-spaces. **Top:** Mona Lisa painting by Leonardo da Vinci, 256×256 pixels. **(a) Left Column:** Gaussian scale-space, top to bottom: $\sigma = 0, 5, 10, 15$. **(b) Middle Column:** Dilation scale-space with quadratic structuring function, $t = 0, 0.25, 1, 4$. **(c) Right Column:** Erosion scale-space with quadratic structuring function, $t = 0, 0.25, 1, 4$.

scheduling and communication problems in abstract project management, for instance. They are also employed to analyse queuing systems, traffic flow and the performance of special array processors. To the best of our knowledge, however, no attempt has been made so far to tackle problems from image analysis with this special algebraic approach.

A large part of our paper is devoted to the analysis of the role of the canonical integral transformations in the before mentioned algebras. First we show that the slope transform in the $(\max, +)$-algebra corresponds to the logarithmic multivariate Laplace transform in the $(+, \times)$-algebra. We study relevant properties of this transform and point out links to convex analysis. This leads to the definition of the so-called Cramer transform as the Legendre-Fenchel transform of the logarithmic Laplace transform. The Cramer transform is well-known in the theory of large deviations in stochastics. In image analysis, it maps standard convolution to $*_e$-convolution, and Gaussians to quadratic functions. This explains why quadratic structuring functions are the morphological equivalent of Gaussian convolution kernels.

Our paper is organised as follows. In Section 2 we introduce the $(\max, +)$ and $(\min, +)$ algebras that will play a fundamental role for the analysis of morphological systems. In Section 3 we show that dilation and erosion are convolutions in these algebras. Connections to convex analysis are explained in Section 4, and the relations between the logarithmic Laplace transform and the Young-Fenchel conjugate are investigated in Section 5. Section 6 is devoted to the Cramer transform which constitutes the explanation for the logarithmic connection between linear and morphological systems. Finally we conclude our paper with a summary in Section 7.

2 The $(\max, +)$- and the $(\min, +)$-Algebra

In the theory of linear systems two algebraic structures play an important role: the $(\max, +)$-algebra $\mathbb{R}_{max}$ and the $(\min, +)$-algebra $\mathbb{R}_{min}$. Formally they emerge from the standard $(+, \cdot)$-algebra $(\mathbb{R}, +, \cdot)$ first by an extension of the real line with either the element $-\infty$ or $+\infty$, second by replacing the addition by a max- or min-operation, and the multiplication by $+$. We have the following table:

name	set	addition	multiplication
standard algebra $\mathbb{R}$	$\mathbb{R}$	$+$	$\times$
$(\max, +)$-algebra $\mathbb{R}_{max}$	$\mathbb{R} \cup \{-\infty\}$	max	$+$
$(\min, +)$-algebra $\mathbb{R}_{min}$	$\mathbb{R} \cup \{+\infty\}$	min	$+$

The algebraic structures $\mathbb{R}_{max}$ and $\mathbb{R}_{min}$ are examples of idempotent semifields. The idempotency has to serve as a substitute for the non-existing inverse w.r.t. the max- or min- operations. For a rather exhaustive amount of details, see [4]. The structural importance of these algebraic structures will become clear in the next section.

3 Convolution Induced by an Algebra

We equip the range of a scalar-valued function with the algebraic structure introduced above, that is, we consider functions

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}_{max} \quad \text{or} \quad f : \mathbb{R}^n \longrightarrow \mathbb{R}_{min}.$$

This gives rise to two analogs to the well-known convolution $*$ stemming from the standard-algebra $(\mathbb{R}, +, \times)$,

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y) \cdot g(y)\, d\,y,$$

for all $x \in \mathbb{R}^n$. The transition from the standard algebra to the other algebras

$$(\mathbb{R}, +, \times) \quad \rightleftharpoons \quad (\mathbb{R} \cup \{+\infty\}, \min, +) \quad \text{or} \quad (\mathbb{R} \cup \{-\infty\}, \max, +)$$

amounts to the replacement of integration (=summation) by taking the infimum or the supremum, and the replacement of multiplication by addition. This leads to the definitions

$$\begin{aligned} (f *_d g)(x) &:= \sup_{y \in \mathbb{R}^n} \big(f(x-y) + g(y) \big) = \sup_{y \in \mathbb{R}^n} \big(f(y) + g(x-y) \big), \\ (f *_e g)(x) &:= \inf_{y \in \mathbb{R}^n} \big(f(x-y) + g(y) \big) = \inf_{y \in \mathbb{R}^n} \big(f(y) + g(x-y) \big). \end{aligned}$$

Hence the morphological operations of dilation $\oplus$ and erosion $\ominus$ as given in [9] or [26] appear as convolutions w.r.t. these algebras:

$$(f \oplus g)(x) = \sup_{y \in \mathbb{R}^n} (f(y) + g(x-y)) = f *_d g(x)\,, \tag{1}$$

$$(f \ominus g)(x) = \inf_{y \in \mathbb{R}^n} (f(y) - g(y-x)) = f *_e \overline{g}(x)\,, \tag{2}$$

with $\overline{g}(x) := -g(-x)$. This explains the notations $*_e$ and $*_d$. Furthermore, the operation $*_e$ coincides exactly with the so-called inf-convolution or epigraphic addition in convex analysis [19, 30], denoted sometimes by $\overset{+}{\vee}$ or $\Box$. Of vital importance in this field is the Legendre–Fenchel transform, which is intimately connected to the slope transform known from morphology. In order to explore this connection, it is worthwhile to pursue a short excursion into convex analysis.

4 Elements of Convex Analysis

Let $\overline{\text{Conv}}\mathbb{R}^n$ be the set of functions $f : \mathbb{R}^n \longrightarrow \mathbb{R} \cup \{+\infty\}$ which are closed convex, that is, convex, lower semicontinuous and finite in at least one point. Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product in $\mathbb{R}^n$. f has an affine minorant iff $f \geq \langle t', \cdot \rangle - c$ for some $(t', c) \in \mathbb{R}^n \times \mathbb{R}$. The convolution $f *_e g$ of two convex functions f, g that have a common affine minorant is again convex. The inf-convolution is an asociative, commutative, order-preserving binary operation. Defining for any subset $A \in \mathbb{R}^n$

$$i_A(x) := \begin{cases} 0 & x \in A\,, \\ +\infty & \text{otherwise}\,, \end{cases} \tag{3}$$

$i_{\{0\}}$ is recognised as the neutral element. This corresponds to the structural element in [9, 23]. In general, closedness is not preserved under inf-convolution.

Definition 1. *The* Legendre-Fenchel transform *or* conjugacy operation *associates with each f with an affine minorant the function f^* defined by*

$$f^*(x) := \sup_{t \in \mathbb{R}^n} \left[\langle t, x \rangle - f(t) \right].$$

Remarkably, $f^* \in \overline{\text{Conv}}\, \mathbb{R}^n$ as soon as f is affinely minorised, regardless of its convexity or closedness [19]. In morphology this operation is a variant of the slope transform [9, 25].

The next theorem states that the function cone $\overline{\text{Conv}}\, \mathbb{R}^n$ is indeed very suitable for this transform: the Legendre-Fenchel transform leaves $\overline{\text{Conv}}\, \mathbb{R}^n$ invariant and is even an involution on this cone. Also of importance are the algebraic properties of this transform with respect to $*_e$:

Theorem 1. (Properties of the Legendre-Fenchel Transform)
If $f, g \in \overline{\text{Conv}}\, \mathbb{R}^n$ then

1. $f^* \in \overline{\text{Conv}}\, \mathbb{R}^n$.
2. *The Legendre-Fenchel transform is its own inverse:* $(f^*)^* = f$.
3. *It maps sums into erosions:* $(f + g)^* = f^* *_e g^*$.
4. *It also maps erosions into sums:* $(f *_e g)^* = f^* + g^*$.

For proofs of these assertions and more detailed results on the properties of conjugation, the reader is referred to [19]. Item 4 deserves a little remark: the conjugacy operation transforms convolution $*_e$ into the sum of the conjugates. It is not an incident that property 4 resembles very much the behaviour of the Laplace transform with respect to the standard convolution $*$ in the $(\mathbb{R}, +, \times)$-algebra. For any function $f : \mathbb{R}^n \longrightarrow [0, +\infty]$, we define the *multivariate Laplace transform* by

$$L[f] : x \longmapsto L[f](x) := \int_{\mathbb{R}^n} e^{\langle x, y \rangle} f(y)\, dy \qquad \text{with } x \in \mathbb{R}^n.$$

Indeed, $*$-convolution of functions is transformed into a multiplication of the Laplace transforms:

$$\begin{aligned} \int_{\mathbb{R}^n} e^{\langle x, y \rangle} \int_{\mathbb{R}^n} f(y - z)\, g(z)\, dz\; dy &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\langle x, y - z \rangle} f(y - z) e^{\langle x, z \rangle} g(z)\, dz\; dy \\ &= \int_{\mathbb{R}^n} e^{\langle x, t \rangle} f(t)\, dt \cdot \int_{\mathbb{R}^n} e^{\langle x, z \rangle} g(z)\, dz\,. \end{aligned}$$

To avoid confusion it should be mentioned that usually the (one-sided) Laplace transform is defined by

$$L_I[f](p) := \int_0^{+\infty} f(t)\, e^{-pt}\, dt$$

with $t \in \mathbb{R}$ and a complex number p. Hence the multivariate integral transform above is an n-dimensional generalisation of the so-called two-sided Laplace transform

$$L_{II}[f](p) := \int_{-\infty}^{+\infty} f(t)\, e^{-pt}\, dt.$$

More details and the close connection between the last two variants of the transform are discussed in [8].

5 A Link between Laplace Transform and Conjugation

Starting from the definition of the conjugacy operation the transition ($\rightleftharpoons$) from the $(\max,+)$-algebra to the $(+,\times)$-algebra entails

$$\begin{aligned} f^*(x) &= \sup_{y\in\mathbb{R}^n} (\langle y,x\rangle - f(y)) = \log \sup_{y\in\mathbb{R}^n} \left(e^{\langle y,x\rangle - f(y)}\right) \\ &\rightleftharpoons \log \int_{\mathbb{R}^n} e^{\langle y,x\rangle - f(y)}\, dy = \log \int_{\mathbb{R}^n} e^{\langle y,x\rangle} \cdot e^{-f(y)}\, dy = \log\, L[e^{-f}](x)\,. \end{aligned}$$

In other words: the conjugate of f interpreted in the context of the $(\max,+)$-algebra corresponds to this logarithmic Laplace transform of e^{-f} in the standard algebra. A logarithmic relation between the two transforms becomes obvious: essentially it traces back to the homomorphism provided by the logarithm:

$$\log(a\cdot b) = \log\, a + \log\, b\,.$$

When compared to Theorem 1 (items 1 and 4), the following proposition emphasises the correspondence between conjugation and logarithmic Laplace transform.

Proposition 1. (Properties of the Logarithmic Laplace Transform) *For any functions $f,g : \mathbb{R}^n \longrightarrow [0,+\infty]$ with $f,g \not\equiv 0$ one has:*

1. *The logarithmic Laplace transform is always convex and lower semicontinuous for non-negative functions:*

$$\log L[f] \in \overline{Conv}\,\mathbb{R}^n\ .$$

2. *Convolutions are maped into sums:*

$$\log L[f * g] = \log L[f] + \log L[g]\,.$$

PROOF: 1. Suppose $0 < \alpha < 1$, then

$$\begin{aligned} L[f](\alpha x_1 + (1-\alpha)x_2) &= \int_{\mathbb{R}^n} e^{\langle \alpha x_1 + (1-\alpha)x_2, y\rangle} f(y)\, dy \\ &= \int_{\mathbb{R}^n} \left(e^{\langle x_1, y\rangle}\right)^{\alpha} \cdot \left(e^{\langle x_2, y\rangle}\right)^{(1-\alpha)} f(y)\, dy \\ &\leq \left(L[f](x_1)\right)^{\alpha} \cdot \left(L[f](x_2)\right)^{1-\alpha} \end{aligned}$$

by Hölder's Inequality with exponents $p = \frac{1}{\alpha}$ and $p' = \frac{1}{1-\alpha}$. Taking the logarithm proves the claimed convexity. The lower-semicontinuity follows directly from Fatou's lemma [5], since

$$\lim_{n \longrightarrow +\infty} x_n = x \qquad \text{implies} \qquad L[f](x) \leq \liminf_{n \longrightarrow +\infty} L[f](x_n),$$

for non-negative f and the fact that the logarithm is increasing and continuous. Property 2 follows directly from the properties of the Laplace transform and the logarithm. □

In this context it is also worth mentioning that there is a continuous transition from the standard $*$-convolution of two positive functions f, g to their $*_e$-convolution, and again the logarithm makes its natural appearance. With reference to the usual Lebesgue norms $\|\cdot\|_p$ with $1 \leq p \leq +\infty$, we define for (strictly) positive functions f, g:

$$(f *_p g)(x) = \|f \cdot g(x - \cdot)\|_p \quad \text{for} \quad 1 \leq p \leq +\infty.$$

On the one hand, for $p = 1$ we regain the well-known convolution: $* = *_1$.
On the other hand, we infer directly from the definitions of the operations $*_d$ that

$$\begin{aligned} (f *_p g)(x) &= \|f \cdot g(x - \cdot)\|_p \\ &\stackrel{p \to +\infty}{\longrightarrow} \|f \cdot g(x - \cdot)\|_\infty \\ &= \exp\left[\log\left[\sup_y (f(y) \cdot g(x-y))\right]\right] \\ &= \exp((\log f) *_d (\log g)(x)). \end{aligned}$$

In a similar fashion we obtain for not necessarily positive functions f, g:

$$\log((e^{-f} *_p e^{-g})(x)) \stackrel{p \to 1}{\longrightarrow} \log((e^{-f} * e^{-g})(x))$$

as well as

$$\begin{aligned} \log((e^{-f} *_p e^{-g})(x)) &\stackrel{p \to \infty}{\longrightarrow} \log\left[\sup_y (e^{-f(y)} \cdot e^{-g(x-y)}\right] \\ &= \log e^{-\inf_y (f(y) + g(x-y))} \\ &= -(f *_e g)(x). \end{aligned}$$

6 The Cramer Transform

The Cramer transform plays a key role in statistics, especially in the theory of large deviations [7, 10]. From a functional point of view, it will allow us to make a connection between the usual convolution $*$, that appears in linear scale-space theory, and the morphological operations $\oplus$ and $\ominus$. This connection makes use of the convolutions $*_d$ and $*_e$. According to its appearance in statistics we will define the Cramer transform for non-negative functions only.

Definition 2. *For functions* $f : \mathbb{R}^n \longrightarrow [0, +\infty]$*, the transform*

$$C[f] := (\log L[f])^*$$

is called Cramer transform.

The reason why this transform is of importance in morphology is illuminated by the following theorem which is a direct consequence of the properties of the Laplace and Legendre-Fenchel transforms.

Theorem 2. (Convolution Theorem for the Cramer Transform)
If f *and* g *are non-negative functions on* $\mathbb{R}^n$*, then*

$$C[f * g] = C[f] *_e C[g] .$$

In view of equations (1) and (2) this entails for nonnegative functions $f, g \not\equiv 0$ the relations

$$-C[f * g] = (-C[f]) \oplus (-\overline{C[g]})$$

and

$$C[f * g] = C[f] \ominus \overline{C[g]} .$$

Let us now discuss some properties of the Cramer transform.
First we observe that, according to Proposition 1, the Cramer transform maps any non-negative function into $\overline{\text{Conv}}\,\mathbb{R}^n$. Hence it follows from Theorem 1 (2) that the conjugate of the Cramer transform is the logarithmic Laplace transform:

$$C^*[f] = \log\, L[f] .$$

Examples of Cramer Transforms

1. Let δ_a denote the Dirac measure in $a \in \mathbb{R}^n$. Then

$$C[\delta_a] = i_a$$

with i_a being defined in (3).

2. The Cramer transform is not additive:

$$C[(1-p)\,\delta_0 + p\,\delta_1](x) = x \cdot \log\left(\frac{x}{p}\right) + (1-x) \cdot \log\left(\frac{1-x}{1-p}\right) + i_{[0,1]}(x).$$

3. The Gauss distributions correlate to quadratic functions with reciprocal "variance": As mentioned in [1, 3] this means for the one dimensional Gaussian with mean μ and variance σ that

$$C\left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{1}{2}(\frac{\cdot - m}{\sigma})^2}\right](p) = \frac{1}{2}\left|\frac{p-m}{\sigma}\right|^2 .$$

This can be extended to the n-variate case of a Gaussian with diagonal covariance matrix. We give a proof of both assertions by first calculating the Laplace transform of a one-dimensional Gaussian with mean $\mu = 0$ and $\sigma^2 > 0$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{t^2}{2\sigma^2}}\, e^{p\cdot t}\, dt \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2\sigma^2}}\, e^{-p\cdot t}\, dt \\
&= \frac{1}{\sqrt{2\pi\sigma^2}} \left(\int_0^{\infty} e^{-\frac{t^2}{2\sigma^2}}\, e^{-p\cdot t}\, dt + \int_0^{\infty} e^{-\frac{t^2}{2\sigma^2}}\, e^{-(-p)\cdot t}\, dt\right) \\
&= \frac{1}{2}\, e^{\frac{1}{2}\sigma^2 p^2} \left(\mathrm{Erfc}(\tfrac{1}{2} p\sqrt{2\sigma^2}\,) + \mathrm{Erfc}(-\tfrac{1}{2} p\sqrt{2\sigma^2}\,)\right) \\
&= e^{\frac{1}{2}\sigma^2 p^2}
\end{aligned}$$

where the complementary error function

$$\mathrm{Erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^{+\infty} e^{-t^2}\, dt$$

is used according to formula 5.41 in [28].
For a Gaussian with mean μ it follows immediately by a simple change of variables that

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{(t-\mu)^2}{2\sigma^2}}\, e^{p\cdot t}\, dt = e^{p\,\mu}\, e^{\frac{1}{2}\sigma^2 p^2} .$$

An n-variate Gaussian distribution with mean vector $\mu \in \mathbb{R}^n$ and diagonal covariance matrix $D = diag(\sigma_1^2, \ldots, \sigma_n^2)$ has the separable density

$$g(y) = \frac{1}{(\sqrt{2\pi})^n \cdot \prod_{i=1}^{n} \sigma_i}\, e^{-\frac{1}{2}\sum_{i=1}^{n} \frac{(y_i-\mu_i)^2}{\sigma_i^2}} .$$

Making use of the results above, Fubini's theorem immediately gives

$$\begin{aligned}
\int_{\mathbb{R}^n} g(y)\, e^{\langle p,y\rangle}\, dy &= \int_{\mathbb{R}^n} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(y_i-\mu_i)^2}{\sigma_i^2}}\, dy \\
&= \prod_{i=1}^{n} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(y_i-\mu_i)^2}{\sigma_i^2}}\, dy_i \\
&= \prod_{i=1}^{n} e^{p_i\mu_i} \cdot e^{\frac{1}{2}p_i^2\sigma_i^2} \\
&= e^{\langle p,\mu\rangle + \frac{1}{2}p^\top Dp}\,.
\end{aligned}$$

Hence we have

$$\log L[g](p) = \langle p,\mu\rangle + \frac{1}{2}p^\top Dp\,.$$

Furthermore a straightforward calculation gives an optimal $p = D^{-1}(s-\mu)$ in $\sup_{p\in\mathbb{R}}\{\langle s,p\rangle - \log\, L[g](p)\}$ which results in

$$\begin{aligned}
(\log L[g])^*(s) &= \left(p \mapsto \langle p,\mu\rangle + \frac{1}{2}p^\top Dp\right)^*(s) && (4)\\
&= \frac{1}{2}\left\langle s-\mu, D^{-1}(s-\mu)\right\rangle && (5)
\end{aligned}$$

for $x \in \mathbb{R}^n$. This result is in complete accordance with the findings in [36].

4. We conclude this set of examples with the numerical evaluation of Cramer transforms of positive, piecewise consant one-dimensional signals f sampled at equidistant points over the interval $[0,1[$. Defining the indicator function $\mathbf{1}_A$ as $\mathbf{1}_A(x) = 1$ if $x \in A$, and 0 otherwise, these signals are of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{[\frac{i-1}{n},\frac{i}{n}[}\,.$$

Their logarithmic Laplace transforms read as

$$\log L[f](s) = \log\left(\frac{1}{s}\sum_{i=1}^{n} \alpha_i \left(e^{s\frac{i}{n}} - e^{s\frac{i-1}{n}}\right)\right)$$

but the corresponding conjugates, that means their Cramer transforms, cannot be computed explicitly. Therefore we depict the graphs of some signals together with their Cramer transforms in Figure 2 below. These results bring to light the very strong smoothing property of the Cramer transform.

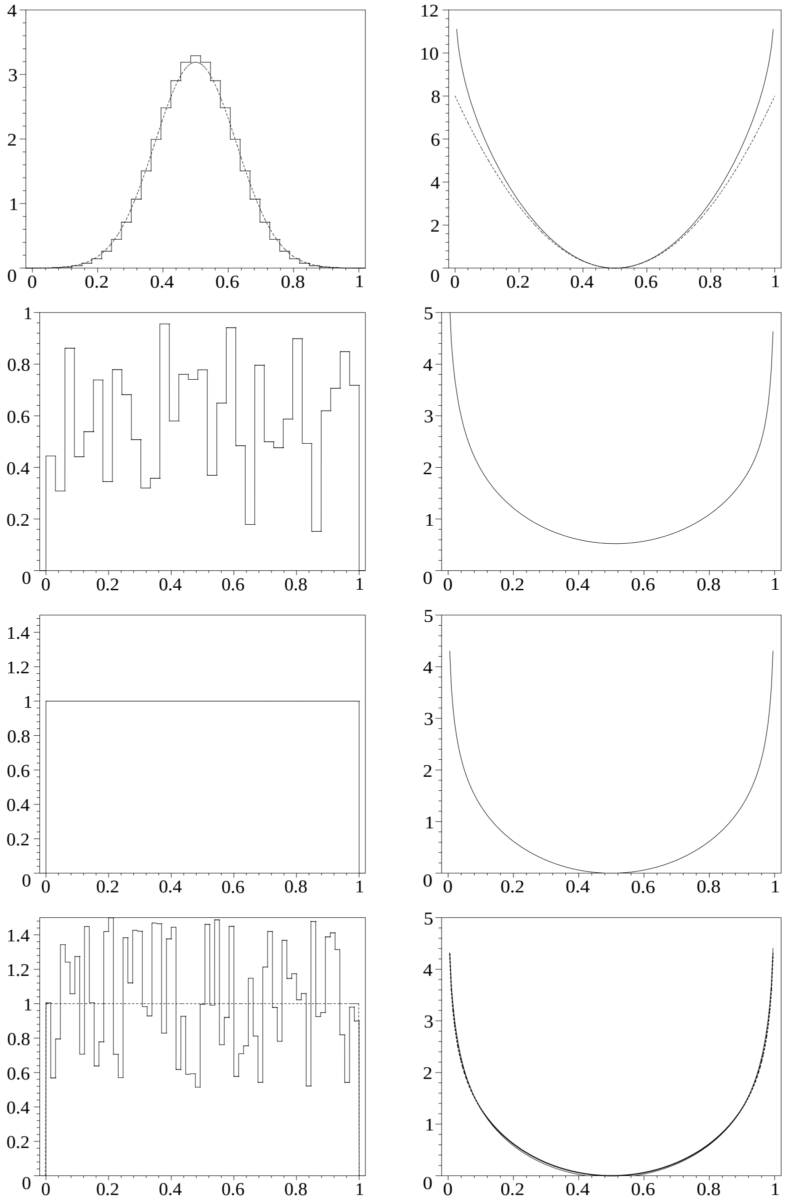

Fig. 2. The smoothing property of the Cramer transform (CT). **Top Row:** A Gaussian and its piecewise constant approximation on 33 subintervals (left) and their CTs (right). **2nd Row:** A random signal, piecewise constant on 33 subintervals (left) and its CT. **3rd Row:** 0-1 signal on the interval [0,1[(left) and its CT. **4th Row:** 0-1 signal on the interval [0,1[with 100% additive uniform noise (left) and its CT vs. the results of the 3rd row.

7 Conclusions

In this paper we have given an explanation for the almost logarithmic connection between linear and morphological systems. This has been achieved by regarding morphological systems as linear systems in appropriate algebras. The link between these algebras and the standard algebra in linear system theory has been established by means of the Cramer transform.

The present article can be regarded as a step towards the unification of linear and morphological scale–space theory on the basis of a general linear system theory in an appropriate algebra. Taking full advantage of this connection may allow to translate results directly from one area to the other. This may trigger a more fruitful interaction of both paradigms that have evolved independently to powerful image processing tools. Finally, a unification within a more general algebraic framework may also help to identify novel image processing approaches that are based on other algebras. These points will be addressed in our future publications.

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