Morphological Operations on Matrix-Valued Images

Bernhard Burgeth, Martin Welk, Christian Feddern, and Joachim Weickert

Mathematical Image Analysis Group
Faculty of Mathematics and Computer Science, Bldg. 27
Saarland University, 66041 Saarbrücken, Germany
{burgeth,welk,feddern,weickert}@mia.uni-saarland.de
http://www.mia.uni-saarland.de

Abstract. The output of modern imaging techniques such as diffusion tensor MRI or the physical measurement of anisotropic behaviour in materials such as the stress-tensor consists of tensor-valued data. Hence adequate image processing methods for shape analysis, skeletonisation, denoising and segmentation are in demand. The goal of this paper is to extend the morphological operations of dilation, erosion, opening and closing to the matrix-valued setting. We show that naive approaches such as componentwise application of scalar morphological operations are unsatisfactory, since they violate elementary requirements such as invariance under rotation. This lead us to study an analytic and a geometric alternative which are rotation invariant. Both methods introduce novel non-component-wise definitions of a supremum and an infimum of a finite set of matrices. The resulting morphological operations incorporate information from all matrix channels simultaneously and preserve positive definiteness of the matrix field. Their properties and their performance are illustrated by experiments on diffusion tensor MRI data.

Keywords: mathematical morphology, dilation, erosion, matrix-valued imaging, DT-MRI

1 Introduction

Modern data and image processing encompasses more and more the analysis and processing of matrix-valued data. For instance, diffusion tensor magnetic resonance imaging (DT-MRI), a novel medical image acquisition technique, measures the diffusion properties of water molecules in tissue. It assigns a positive definite matrix to each voxel, and the resulting matrix field is a valuable source of information for the diagnosis of multiple sclerosis and strokes [13]. Matrix fields also make their natural appearance in civil engineering and solid mechanics. In these areas inertia, diffusion and permittivity tensors and stress-strain relationships are an important tool in describing anisotropic behaviour. In the form of the so-called structure tensor (also called Förstner interest operator, second moment matrix or scatter matrix) [7] the tensor concept turned out to be of great value

in image analysis, segmentation and grouping [9].
So there is definitely a need to develop tools for the analysis of such data since anybody who attempts to do so, is confronted with the same basic tasks as in the scalar-valued case: How to remove noise, how to detect edges and shapes, for example.

Image processing of tensor fields is a very recent research area, and a number of methods consists of applying scalar- and vector-valued filters to the components, eigenvalues or eigenvectors of the matrix field. Genuine matrix-valued concepts with channel interaction are available for nonlinear regularisation methods and related diffusion filters [17, 18], for level set methods [6], median filtering [19] and homomorphic filters [4]. To our knowledge, however, extensions of classical morphology to the matrix setting have not been considered so far.

Our paper aims at closing this gap by offering extensions of the fundamental morphological operations dilation and erosion to matrix-valued images. Mathematical morphology has been proven to be useful for the processing and analysis of binary and greyscale images: Morphological operators and filters perform noise suppression, edge detection, shape analysis, and skeletonisation in medical and geological imaging, for instance [15]. Even the extension of concepts of scalar-valued morphology to vector-valued data such as colour images, is by no means straightforward. The application of standard scalar-valued techniques to each channel of the image independently, that means component-wise performance of morphological operations, might lead to information corruption in the image, because, in general, these components are strongly correlated [1, 8]. Numerous attempts have been made to develop satisfying concepts of operators for colour morphology. The difficulty lies in the fact that the morphological operators rely on the notion of infimum and supremum which in turn requires an appropriate ordering of the colours, i.e. vectors in the selected vector space. However, there is no generally accepted definition of such an ordering [2, 16, 12]. Different types of orderings such as marginal or reduced ordering [2] are reported to result in an unacceptable alteration of colour balance and object boundaries in the image [5], or in the existence of more than one supremum (infimum) creating ambiguities in the output image [12]. These are clear disadvantages for many applications. In connection with noise suppression morphological filters based on vector ranking concepts [2] have been developed [11, 5]. In [3] known connections between median filters, inf-sup operations and geometrical partial differential equations [10] have been extended from the scalar to the vectorial case.

In any case, the lack of a generally suitable ordering on vector spaces is a very severe hindrance in the development of morphological operators for vector-valued images. Surprisingly the situation in the matrix-valued setting is more encouraging since we have additional analytic-algebraic or geometric properties of the image values at our disposal: (a) Unlike in the vectorial setting one can multiply matrices, define polynomials and even can take roots of matrices. (b) Real symmetric, positive definite matrices can be graphically represented by ellipses (2×2-matrices) or ellipsoids (3×3-matrices) in a unique way. However, there is also the burden of additional conditions that have to be fulfilled by the morphological operations to be defined: They have to be rotationally invariant and they must preserve the positive definiteness of the matrix field as well, since applications such as DT-MRI create such data sets. In this paper we will exploit the analytic-algebraic property (a) and the geometric property (b) by introducing novel notions for the supremum/infimum of a finite set of matrices. These notions are rotationally invariant and preserve positive definiteness.

Interestingly, already the requirement of rotational invariance rules out the straightforward component-wise approach: Consider for example

$$A_1 := \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \; A_2 := \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \; S := \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Here, S is the componentwise supremum of A_1, A_2. Rotating A_1 and A_2 by 90 degrees and taking again the componentwise supremum yields

$$A_1' = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}, \; A_2' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \; S' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$$

where S' is clearly not obtained by rotating S. This counterexample shows that it is not obvious how to design reasonable extensions of morphological operations to the matrix-valued setting.

The structure of our paper is as follows: In the next section we give a very brief review of the basic greyscale morphological operations. Then we establish novel definitions of the crucial sup- and inf-operations in the vector valued case via the *analytic-algebraic* approach and investigate some of their properties in Section 3. Alternatively, in Section 4 we develop new definitions for the sup- and inf-operations starting from a *geometric* point of view. Section 5 is devoted to experiments where the two methodologies are applied to real DT-MRI images. Concluding remarks are presented in Section 6.

2 Mathematical Morphology in the Scalar Case

In greyscale morphology an image is represented by a scalar function $f(x, y)$ with $(x, y) \in \mathbb{R}^2$. The so-called *structuring element* is a set B in $\mathbb{R}^2$ that determines the neighbourhood relation of pixels with respect to a shape analysis task. Greyscale *dilation* $\oplus$ replaces the greyvalue of the image $f(x, y)$ by its supremum within a mask defined by B:

$$(f \oplus B)\,(x, y) \;:=\; \sup\,\{f(x-x',\, y-y') \mid (x', y') \in B\},$$

while *erosion* $\ominus$ is determined by

$$(f \ominus B)\,(x, y) \;:=\; \inf\,\{f(x+x',\, y+y') \mid (x', y') \in B\}.$$

The *opening* operation, denoted by $\circ$, as well as the closing operation, indicated by the symbol $\bullet$, are defined via concatenation of erosion and dilation:

$$f \circ B := (f \ominus B) \oplus B \quad \text{and} \quad f \bullet B := (f \oplus B) \ominus B\,.$$

These operations form the basis of many other processes in mathematical morphology [14, 15].

3 Model 1: An Analytic Definition of Dilation and Erosion for Matrix-Valued Images

The decisive step in defining morphological dilation and erosion operations for matrix-valued data is to find a suitable notion of supremum and infimum of a finite set of positive definite matrices. For positive real numbers $a_1, \ldots, a_k$, $k \in \mathbb{N}$, there is a well-known connection between their modified p-mean and their supremum:

$$\lim_{p \to +\infty} \left(\sum_{i=1}^{k} a_i^p \right)^{\frac{1}{p}} = \sup\{a_1, \ldots, a_k\} \,. \tag{1}$$

A completely analogous relation holds also for the infimum with the difference that p now tends to $-\infty$:

$$\lim_{p \to -\infty} \left(\sum_{i=1}^{k} a_i^p \right)^{\frac{1}{p}} = \inf\{a_1, \ldots, a_k\} \,. \tag{2}$$

That means, the p-means can serve as a substitute for the supremum (infimum) if p is large. The idea is now to replace the positive numbers in the above relation by their matrix generalisations, the positive definite matrices $A_1, \ldots, A_k$. However, to this end we have to define the p-th root of a positive definite $(n \times n)$-matrix A . We know from linear algebra that there exists an orthogonal $(n \times n)$-matrix V (which means $V^\top V = V V^\top = I$, with unit matrix I) such that

$$A = V \cdot \mathrm{diag}(\alpha_1, \ldots, \alpha_n) \cdot V^\top \,, \tag{3}$$

where the expression in the center on the right denotes the diagonal matrix with the positive eigenvalues $\alpha_1, \ldots, \alpha_n$ of A as entries on the diagonal. Now taking the p-th root of a matrix is achieved by taking the p-th root of the eigenvalues in decomposition (3):

$$A^{\frac{1}{p}} := V \mathrm{diag}(\alpha_1^{\frac{1}{p}}, \ldots, \alpha_n^{\frac{1}{p}}) V^\top \,.$$

Note that the p-th power A^p can be calculated in this manner as well. Hence we can give meaning to the expression $\left(\sum_{i=1}^{k} A_i^p \right)^{\frac{1}{p}}$ and can define new matrices $\sup\{A_1, \ldots, A_k\}$ and $\inf\{A_1, \ldots, A_k\}$ via the limits of their modified p-mean for $p \to \pm\infty$:

Definition 1. *The* supremum *and* infimum *of a set of positive definite matrices* $A_1,\ldots,A_k$ *are defined as*

$$\sup\{A_1, \ldots, A_k\} := \lim_{p \to +\infty} \left(\sum_{i=1}^{k} A_i^p \right)^{\frac{1}{p}} \,, \tag{4}$$

$$\inf\{A_1, \ldots, A_k\} := \lim_{p \to -\infty} \left(\sum_{i=1}^{k} A_i^p \right)^{\frac{1}{p}} \,. \tag{5}$$

With this definition, taking the supremum is a rotationally invariant operation, i.e. $\sup\{UA_1U^\top,\dots,UA_kU^\top\} = U\cdot\sup\{A_1,\dots,A_k\}\cdot U^\top$ for any orthogonal $(n\times n)$-matrix U. This may be seen as follows. Since $\sum_{i=1}^k A_i^p$ is positive definite, there exist an orthogonal matrix V and a diagonal matrix D with $\sum_{i=1}^k A_i^p = VDV^\top$. As a consequence we obtain

$$\begin{aligned}\Big(\sum_{i=1}^k (UA_iU^\top)^p\Big)^{\frac{1}{p}} &= \Big(\sum_{i=1}^k UA_i^pU^\top\Big)^{\frac{1}{p}} = \Big(U\big(\sum_{i=1}^k A_i^p\big)U^\top\Big)^{\frac{1}{p}}\\ &= \Big(UVDV^\top U^\top\Big)^{\frac{1}{p}} = UVD^{\frac{1}{p}}V^\top U^\top = U\Big(\sum_{i=1}^k A_i^p\Big)^{\frac{1}{p}}U^\top\,,\end{aligned}$$

where we have used the facts that $U^\top U = I$ and UV is also orthogonal. Therefore the p-th mean is rotationally invariant for all values of p, and hence also in the limits $p\to\pm\infty$.

Furthermore the p-th mean (and in the limit also supremum and infimum) inherits the positive definiteness of its arguments: Positive definiteness is a property stable under addition, and is also characterised by the positivity of the eigenvalues. By construction the p-th power A^p and the p-th root $A^{\frac{1}{p}}$ have positive eigenvalues whenever A has. Hence taking the p-th mean for any $p\in\mathbb{N}$ preserves positive definiteness.

For practical computations we will put p to a sufficiently large number, say 10 or 20, such that the resulting matrices can be considered as reasonable approximations to the supremum resp. infimum of $A_1,\dots,A_k$.

Alternatively, the limiting matrix $M := \sup\{A_1,\dots,A_k\}$ can also be obtained directly from the eigenvalues and eigenvectors of the given set of matrices $A_1,\dots,A_k$. The largest eigenvalue and corresponding eigenvector are directly adopted for M. In the 2×2 case, the eigenvector system of M is already determined by this condition. The remaining eigenvalue of M is exactly the largest eigenvalue from the given set of matrices that corresponds to an eigenvector different from that of the largest eigenvalue – in general, the second largest eigenvalue from the given set. A similar statement holds in higher dimensions. Moreover, replacing largest by smallest eigenvalues, a characterisation of infima is obtained. We sketch the proof for suprema of 2×2 matrices. Note first that the sum $\sum A_i^p$ does not change if every matrix A_i is replaced by the two rank-one matrices $\lambda_1 v_1 v_1^\top$ and $\lambda_2 v_2 v_2^\top$ corresponding to the eigenvalue-eigenvector pairs (λ_1, v_1) and (λ_2, v_2) of A_i. Let now Λ be the largest eigenvalue from the given set of matrices, and λ the second-largest one in the sense described above. Without loss of generality, assume that the eigenvector of Λ is $(1,0)^\top$; the normalised eigenvector for λ is some $(c,s)^\top$, $c^2+s^2=1$. Since the contributions of all smaller eigenvalues and corresponding eigenvectors vanish in the limit $p\to+\infty$, all we have to prove is that the p-mean

$$M_p := \left(\Lambda^p\begin{pmatrix}1\\0\end{pmatrix}(1,0) + \lambda^p\begin{pmatrix}c\\s\end{pmatrix}(c,s)\right)^{\frac{1}{p}} = \begin{pmatrix}\Lambda^p+\lambda^p s^2 & \lambda^p cs\\ \lambda^p cs & \lambda^p c^2\end{pmatrix}^{\frac{1}{p}}$$

tends to $\mathrm{diag}(\Lambda, \lambda)$ for $p \to +\infty$. We introduce the abbreviations $D_p := \Lambda^{2p} - 2\Lambda^p \lambda^p (c^2 - s^2) + \lambda^{2p}$ and $E_p := \Lambda^p - \lambda^p (c^2 - s^2)$. Then we can express the eigenvalues of M_p by $\left(\frac{1}{2}\left(\Lambda^p + \lambda^p \pm \sqrt{D_p}\right)\right)^{1/p}$ which tend to Λ and λ for $p \to +\infty$. An eigenvector for the larger eigenvalue is given by $\left(\sqrt{\sqrt{D_p} + E_p}, \sqrt{\sqrt{D_p} - E_p}\right)^\top$, which encloses with $(1,0)^\top$ the angle φ_p that satisfies $\tan^2 \varphi_p = \frac{\sqrt{D_p}+E_p}{\sqrt{D_p}-E_p}$. Since the latter expression tends to 0 for $p \to +\infty$ if $\lambda < \Lambda$, we have that the limiting matrix is diagonal as claimed. In case $\lambda = \Lambda$ we have already that $\lim\limits_{p \to +\infty} M_p$ is diagonal because of the eigenvalues. This completes the proof.

With the supremum and infimum operations at our disposal we can apply the definitions of the basic morphological operations dilation, erosion, opening and closing to matrix-valued images essentially verbatim.

4 Model 2: A Geometric Definition of Dilation and Erosion for Matrix-Valued Images

We present now an alternative framework of dilation and erosion for positive definite symmetric matrices. To this end we remark that a positive definite symmetric $n \times n$ matrix A corresponds to a quadratic form $Q(x) = x^\top A^{-2} x$, $x \in \mathbb{R}^n$. The ellipsoid $x^\top A^{-2} x = 1$ centered around 0 is an isohypersurface of Q. This ellipsoid has a natural interpretation in the context of diffusion tensors: Assuming that a particle is initially located in the origin and is subject to the diffusivity A, then the ellipsoid encloses the smallest volume within which this particle will be found with some required probability after a short time interval. The directions and lengths of the principal axes of the ellipsoid are given by the eigenvectors and corresponding eigenvalues of A, respectively. By including degenerate ellipsoids this description is easily extended to all positive definite symmetric matrices. Then each positive definite matrix A is represented by the image $A\mathcal{B}$ of the unit ball $\mathcal{B} \subseteq \mathbb{R}^n$ under multiplication with A.

Geometric inclusion constitutes a natural semi-order for ellipsoids which leads directly to a semi-order for positive definite matrices.

Definition 2. *Let A, B be positive definite matrices. We define that $A \subseteq B$ if and only if $A\mathcal{B} \subseteq B\mathcal{B}$ where $\mathcal{B}$ is the unit ball in $\mathbb{R}^n$.*

In the language of diffusion tensors $A \subseteq B$ means that for particles evolving under diffusivities A and B, the ellipsoid in which the first one is most probably found is completely contained in the corresponding ellipsoid for the second.

In the light of this semi-order, it makes sense to define the supremum of a set of positive definite matrices as a minimal element (in some sense) among all matrices that are greater or equal to all given matrices. Since, however, the $\subseteq$ semi-order itself is not sufficient to determine such a minimal element, we need an additional criterion. Therefore we introduce a second relation $\preccurlyeq$ which is compatible to the first one in the sense that $A \subseteq B$ always implies $A \preccurlyeq B$.

Definition 3. *Let A, B be as above. We define that $A \preccurlyeq B$ if the ordered sequence $\lambda_1(A) \geq \ldots \geq \lambda_n(A) \geq 0$ of the eigenvalues of A is lexicographically smaller or equal to the corresponding sequence $\lambda_1(B) \geq \ldots \geq \lambda_n(B) \geq 0$ of B, i.e. if there exists an index j, $1 \leq j \leq n+1$ such that $\lambda_i(A) = \lambda_i(B)$ for all $i < j$, and $\lambda_j(A) < \lambda_j(B)$ if $j \leq n$.*

Note that $\preccurlyeq$ is not a semi-order in strict sense because it does not allow to distinguish between a matrix and rotated versions of it. We can now define the supremum of a set of positive definite matrices.

Definition 4. *Let $A_1, \ldots, A_k$ be positive definite symmetric matrices. We define*

$$\sup\{A_1, \ldots, A_k\} := S$$

where S is chosen such that $A_i \subseteq S$ for $i = 1, \ldots, k$, and $S \preccurlyeq Y$ for each Y satisfying $A_i \subseteq Y$ for $i = 1, \ldots, k$.

By reverting all occurrences of $\subseteq$ and $\preccurlyeq$ we obtain an analog definition that introduces the infimum as a $\preccurlyeq$-maximal element in the set of all matrices which are inferior to all given matrices w.r.t. $\subseteq$. The positive definiteness of the so defined supremum and infimum is obvious from the definition, as is the rotational invariance. A closer look shows that if all A_i are positive definite, one has also that the supremum of the inverses A_i^{-1} is the inverse of the infimum of the A_i and vice versa. This is in analogy to the definitions based on the p-mean.
Since it is not obvious how to compute the supremum of a given set $\{A_1, \ldots, A_k\}$ of tensors, we shall now briefly derive the necessary formulae in the case of 2×2 matrices. Assume that $\varLambda$ is the largest eigenvalue of all given matrices, and that $(1, 0)^\top$ is the corresponding eigenvector. Then this eigenvalue–eigenvector pair is also one for the desired supremum matrix S. We have therefore $S = \operatorname{diag}(\varLambda, \lambda)$ where $\lambda \leq \varLambda$ is still to be determined. The decisive constraint for λ is that for all given matrices A_i, the images of the unit disk under $S^{-1}A_i$ must be contained in the unit disk. For a single matrix $A_i = \left(\begin{smallmatrix} a & c \\ c & b \end{smallmatrix}\right)$ this condition comes down to

$$\sqrt{(a\varLambda^{-1} + b\lambda^{-1})^2 + (c\varLambda^{-1} - c\lambda^{-1})^2} + \sqrt{(a\varLambda^{-1} - b\lambda^{-1})^2 + (c\varLambda^{-1} + c\lambda^{-1})^2} \leq 2$$

(note that it is insufficient to consider only the largest eigenvalue of $S^{-1}A$ since this matrix is in general asymmetric!). From this inequality we obtain by squaring twice, re-arranging terms and finally taking the root again that

$$\lambda \geq \sqrt{\frac{(b^2 + c^2)\varLambda^2 - (ab - c^2)^2}{\varLambda^2 - a^2 - c^2}}. \tag{6}$$

Iterating over all A_i one finds the smallest λ which satisfies all the conditions simultaneously. Dismissing the condition that the eigenvector corresponding to $\varLambda$ is $(1, 0)^\top$, the eigenvector system of S is still determined by this eigenvector. One only has to rotate all matrices A_i using this eigenvector system before computing the bounds for λ. This completes the algorithm in the 2×2 case.

Extension of the algorithm to 3×3 and larger matrices works by considering suitable sets of 2-dimensional sections to which the above formulae can be applied. That it is sufficient to consider sections is a consequence of the following observation: Given an ellipsoid centered at the origin and a point outside of it, then the smallest ellipsoid centered at 0 that encloses both is tangent to the first ellipsoid along an ellipse (or, in higher dimensions, an ellipsoid of next smaller dimension). Repeating the above reasoning for the case of erosions, it becomes clear that the smallest eigenvalue Λ of S and corresponding eigenvector are directly obtained as the smallest eigenvalue and corresponding eigenvector of one of the A_i. By analog considerations as above one derives upper bounds for the remaining eigenvalue λ (which is now the larger one). Surprisingly, the bounds are the same as in (6), only the relation sign is reverted to $\leq$.

Revisiting the p-mean approach from the viewpoint of the current section, one sees that the p-mean supremum M of a set $\{A_1, \ldots, A_k\}$ satisfies $A_i \subseteq M$ for all $i = 1, \ldots, k$, and has the same largest eigenvalue and corresponding eigenvector as the supremum S defined here. However, in generic cases $S \subseteq M$ and $S \neq M$ hold, and the eigenvalues of M except the largest one exceed the corresponding ones of S. Thus, M is in general not a minimal element in the set of all Y with $A_i \subseteq Y$ for all i. Analog considerations apply to the p-mean infimum.

5 Experimental Results

In order to illustrate the differences between model 1 and 2, we have computed their behaviour on two ellipses. This is depicted in Figure 1. We observe that model 1 tends to reduce the eccentricity of the ellipses, whereas the more complicated model 2 is constructed in such a way that it corresponds exactly with our geometric intuition.

As a real-world test image we use a DT-MRI data set of a human brain. We have extracted a 2-D section from the 3-D data. The 2-D image consists of four quadrants which show the four tensor channels of a 2×2 matrix. Each channel has a resolution of 128×128 pixels. The top right channel and bottom left channel are identical since the matrix is symmetric. Model 1 is always shown on the left side, model 2 always on the right side. All images are generated using a disk-shaped stencil of radius $\sqrt{5}$. As mentioned in section 2 the simplified algorithm has been used for model 1. Figure 2 shows the results of the erosion and dilation filter on tensor-valued data for both models. Corresponding filters give very similar results. The main difference, as mentioned before, is the tendency of model 1 to reduce direction information faster than model 2 does (see also Figure 4).

This results in a slightly higher contrast in the images in model 2. A number of dark spots that appear in the main diagonal parts of the eroded images indicate violations of the positive definiteness condition. Due to measurement errors, these are already present in the original data set but are widened by erosion.

The experiments for opening and closing can be seen in Figure 3. They confirm the previous impression: There is a high similarity between the test results from model 1 and model 2, the main difference being in the off diagonal where the higher contrast of model 2 is noticeable again.

The main goal, to create a filter for tensor valued erosion and dilation (and the derived opening and closing) which is similar to the scalar case, has been

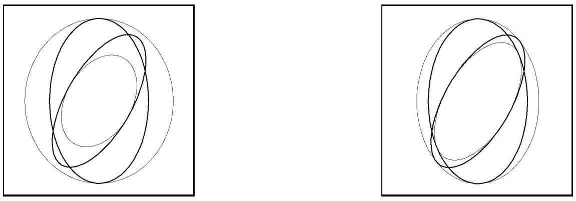

Fig. 1. *Left:* Ellipses representing two positive definite matrices (thick lines), their supremum and infimum (thin lines) according to model 1. *Right:* Same with model 2.

achieved by both models. Whereas model 2 shows somewhat better results in the experiments, model 1 has the advantage of being simpler to implement by using the method based on the two largest eigenvalues.

6 Conclusions

In this paper we have extended fundamental concepts of mathematical morphology to the case of matrix-valued data. Based on two alternative approaches, definitions for supremum and infimum of a set of positive definite symmetric matrices were given. One set of definitions relies on the property of scalar-valued p-means that they tend to the maximum and minimum of their argument sets for $p \to \pm\infty$; supremum and infimum of matrix sets are constructed by an analogous limiting procedure. The second approach combines geometrical and analytical tools to construct suprema and infima as minimal and maximal elements of sets of upper resp. lower bounds of the given matrix set. Each of the two approaches enables the generalisation of morphological dilation, erosion and the further operations composed from these, like opening and closing. In the experimental part, we have implemented the different concepts and evaluated them on diffusion tensor data. Our future investigation will include a more detailed study of the morphological framework built on these operations.

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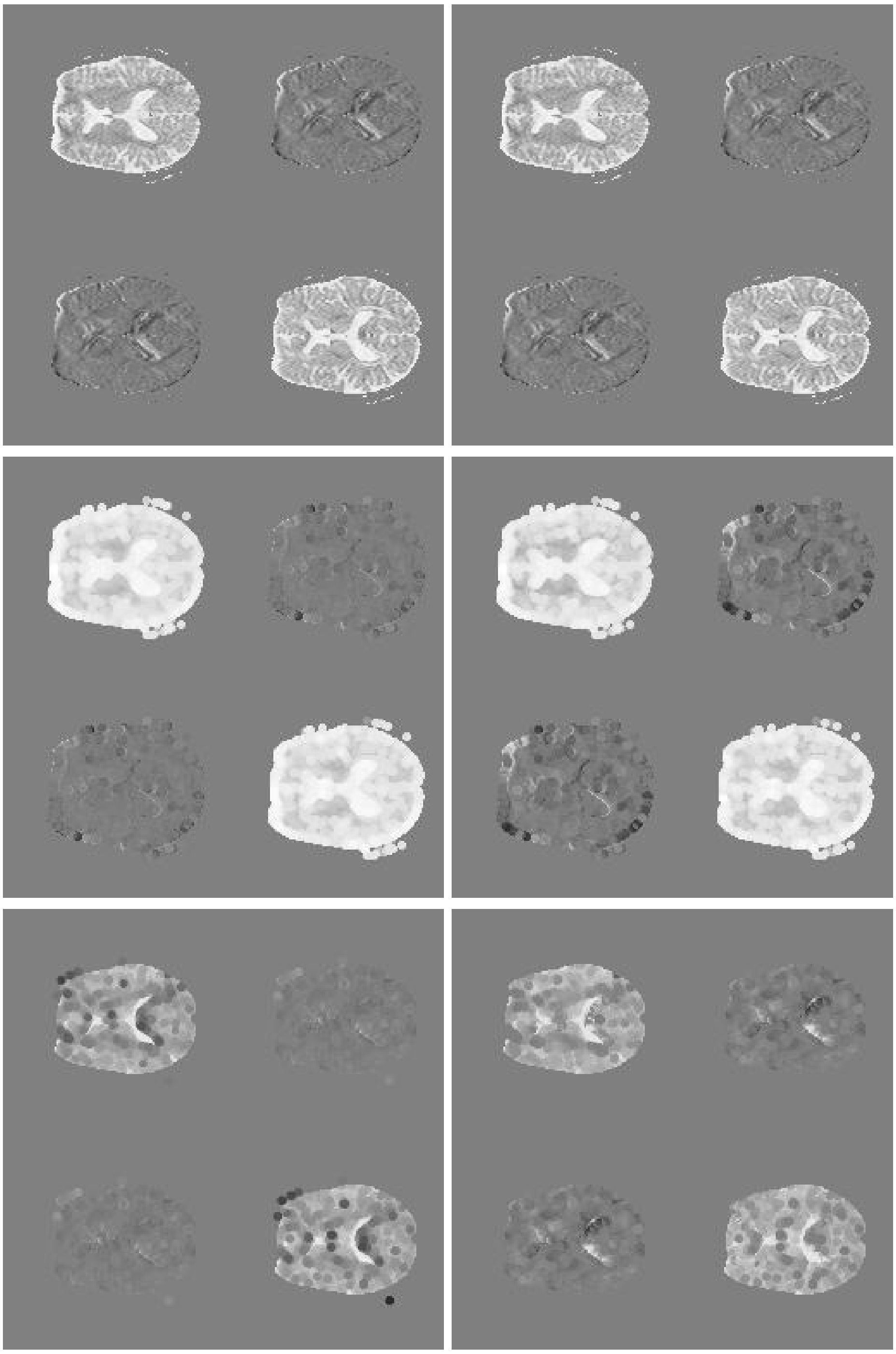

Fig. 2. Tensor-valued dilation and erosion. *Left column, from top to bottom:* Original tensor image of size 128×128 per channel, dilation model 1 with disk-shaped stencil of radius $\sqrt{5}$, erosion model 1 with disk-shaped stencil of radius $\sqrt{5}$. *Right column, from top to bottom:* Same with model 2.

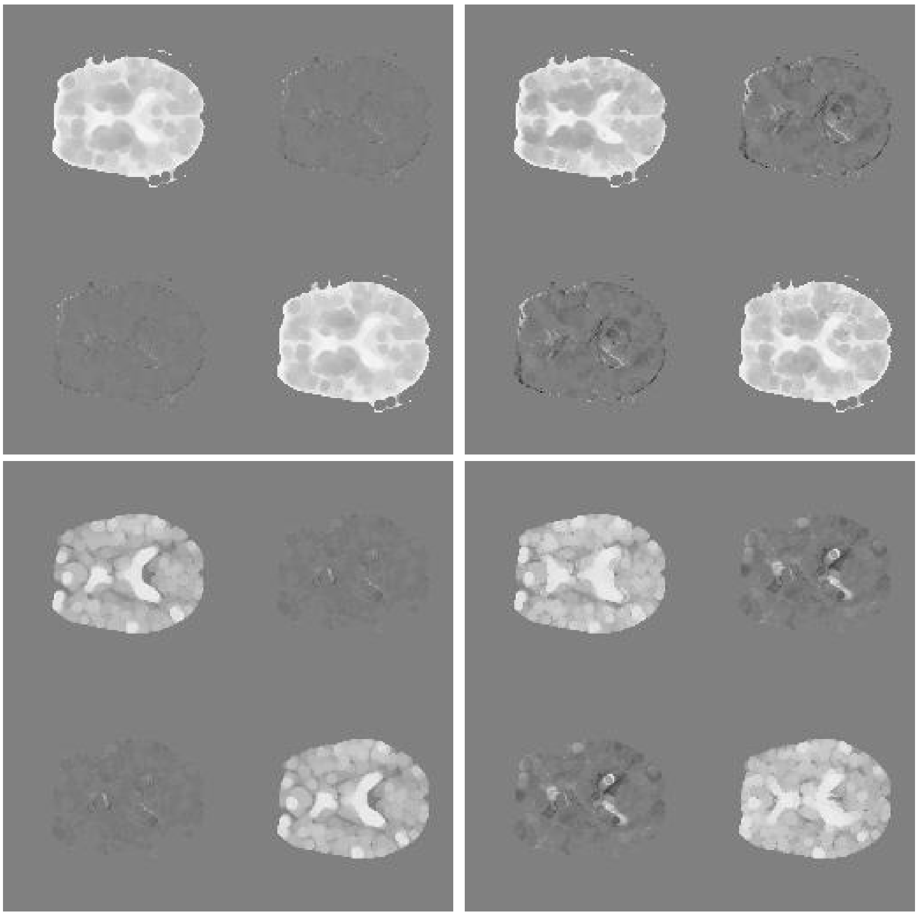

Fig. 3. Tensor-valued opening and closing. *Left column, from top to bottom:* Closing model 1 with disk-shaped stencil of radius $\sqrt{5}$, opening model 1 with disk-shaped stencil of radius $\sqrt{5}$ of the original tensor image depicted in Fig. 2. *Right column, from top to bottom:* Same with model 2.

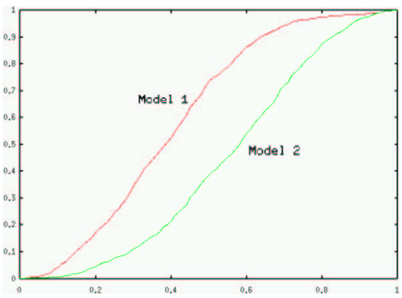

Fig. 4. The distribution of numerical eccentricities $e = \sqrt{1 - \lambda_2^2/\lambda_1^2}$ in the dilated images from Fig. 2.

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