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Variational Optical Flow Estimation

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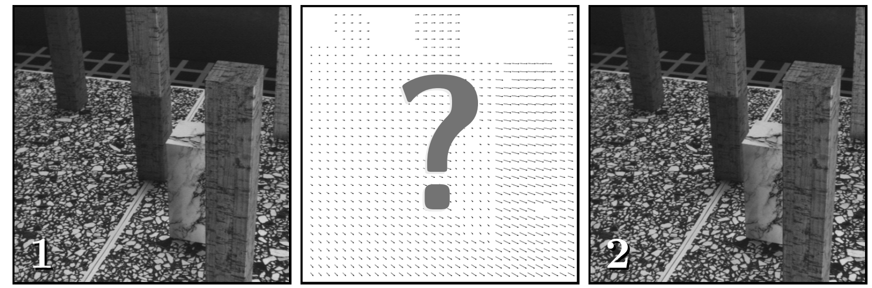
Introduction (1)

MIA 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33

What is the Optical Flow Problem?

- **Given**
  - two or more frames of an image sequence
- **Wanted**
  - displacement field between two consecutive frames → **optical flow**



Introduction (2)

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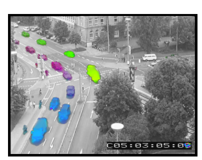
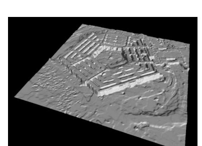
What is Optical Flow Good for?

- **Extraction of Motion Information**
  - robot navigation/driver assistance
  - surveillance/tracking
  - action recognition

- **Processing of Image Sequences**
  - video compression
  - ego motion compensation



- **Related Correspondence Problems**
  - stereo reconstruction
  - structure-from-motion
  - medical image registration



Introduction (3)

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Why Variational Methods?

- **Advantages w.r.t. Modeling** ✓
  - transparent modeling
  - formulation as optimization problem
- **Advantages w.r.t. Computation** ✓
  - unique minimizer and well-posedness
  - real-time capable numerical schemes
- **Advantages w.r.t. Quality** ✓
  - dense flow fields with sub-pixel precision
  - most accurate results in the literature

These are the reasons why variational methods are so successful !

Outline of this Tutorial

- **Part I: Variational Basics** (*Andrés Bruhn*)
  - Continuous modeling
  - Method of Horn and Schunck
- **Part II: Modeling Aspects** (*Thomas Brox*)
  - Motion discontinuities
  - Robust data terms
  - Large displacements
- **Part III: Efficient Numerics** (*Andrés Bruhn*)
  - Improved non-hierarchical solvers
  - Linear and nonlinear multigrid
  - Implementations on parallel hardware

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PART I
Variational Basics

Contents

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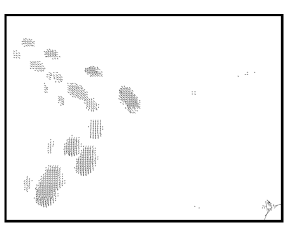
Continuous Modeling

- **Given**
  - continuous image sequence $I_0(x,y,t)$ — location $(x,y) \in \Omega$, time $t \in [0,T]$
- **Wanted**
  - interframe displacement field $\mathbf{w}(x,y,t) = \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \\ 1 \end{pmatrix}$ → **optical flow**

$I_0(x,y,t)$



$\mathbf{w}(x,y,t)$

$I_0(x,y,t+1)$

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Standard Preprocessing

- *Idea*: In order to reduce the influence of noise and outliers, we convolve I_0 with a Gaussian K_σ of mean $\mu = 0$ and standard deviation σ

$$I(x,y,t) = K_\sigma * I_0(x,y,t)$$

  - image sequence becomes infinitely many times differentiable, i.e. $I \in \mathcal{C}^\infty$
  - allows to estimate larger displacements due to the blurring of objects

Important for methods that rely on the computation of image derivatives!

The Gray Value Constancy Assumption

- *Idea:* In order to retrieve corresponding pixels in subsequent frames, we assume that their gray value does not change over time:

$$I(x+u, y+v, t+1) - I(x, y, t) = 0\,.$$

The Linearized Gray Value Constancy Assumption

- *Idea:* If u and v are small and I is sufficiently smooth, one may **linearize** this constancy assumption via a first-order Taylor expansion around the point (x, y, t):

$$I(x+u, y+v, t+1) \approx I(x,y,t) + I_x(x,y,t)u + I_y(x,y,t)v + I_t(x,y,t)1$$

$$\rightarrow I_x u + I_y v + I_t = 0\,.$$

This constraint is the **brightness constancy constraint equation (BCCE)**. In general such constraints on the flow are called optical flow constraints (OFCs).

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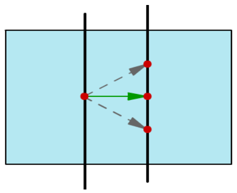
The Aperture Problem

- The BCCE provides only one equation for determining two unknowns
- Ill-posed problem with infinitely many solutions
- Only the flow component in direction of the image gradient can be computed, the so-called **normal flow**:

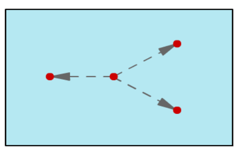
$$(u,v)_{\mathrm{n}}^{\top} = \frac{-I_t}{|\nabla f|} \frac{\nabla I}{|\nabla I|}\,.$$

- This problem is referred to as the **aperture problem**. It can be illustrated as



Case I

$|\nabla I| \neq 0 \rightarrow$ Aperture problem



Case II

$|\nabla I| = 0 \rightarrow$ No estimation possible

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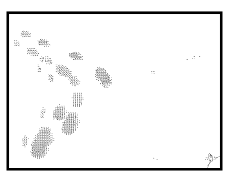
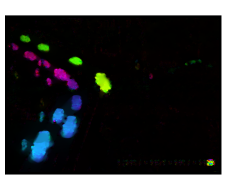
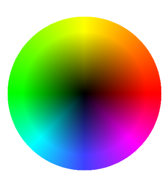
Intermezzo I - How to Visualize Optical Flow Fields?

- *Vector Plot:* Subsample vector field and use arrows for visualization



- *Color Plot:* Visualize direction as color and magnitude as brightness





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Intermezzo II - How to Measure the Quality of Optical Flow Fields?

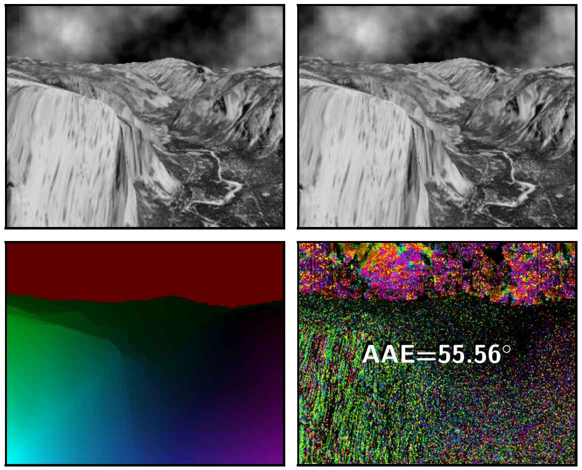
- *Given*: estimated flow field $\mathbf{w}^{\mathrm{e}}$ and ground truth flow field $\mathbf{w}^{\mathrm{t}}$
- *Spatiotemporal Average Angular Error (AAE)*:
  - Consider angle and magnitude by using the **spatiotemporal angle**

$$AAE = \frac{1}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} \arccos\left(\frac{\mathbf{w}_{i,j}^{\mathrm{t}}}{|\mathbf{w}_{i,j}^{\mathrm{t}}|}^{\top} \frac{\mathbf{w}_{i,j}^{\mathrm{e}}}{|\mathbf{w}_{i,j}^{\mathrm{e}}|} \right)\,.$$

- *Average Endpoint Error (AEE)*:
  - Consider the **Euclidean distance** between the vectors

$$AEE = \frac{1}{NM} \sum_{i=1}^{N} \sum_{j=1}^{M} |\mathbf{w}_{i,j}^{\mathrm{t}} - \mathbf{w}_{i,j}^{\mathrm{e}}|\,.$$

How Accurate is the Normal Flow?



Results for the Yosemite Sequence with clouds (L. Quam). **(a) Upper Left:** Frame 8. **(b) Upper Right:** Frame 9. **(c) Lower Left:** Ground truth. **(d) Lower Right:** Normal flow.

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Variational Optical Flow Computation

What is a Functional?

- *Known:* A function maps an **input value** to an output value, e.g.

$$f(x, y) = x^2 + y^2 \ .$$

- *New:* A functional maps an **input function** to an output value, e.g.

$$E(f(x,y)) = \frac{1}{|\Omega|} \int_\Omega f(x,y)\, dx\, dy \ .$$

- *Remarks:* Functionals
  - can be used to rate the quality of a function w.r.t. certain assumptions
  - form the basis of variational optical flow methods

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Principle of Variational Optical Flow Methods

- *Idea:* Compute displacement field as **minimizer** of a suitable energy functional:

$$E(u,v) = \int_\Omega \underbrace{D(u,v)}_{\text{data term}} + \alpha \underbrace{S(u,v)}_{\text{smoothness term}} \, dx\, dy \ .$$

  - **data term** $D(u,v)$ penalizes deviations from constancy assumptions
  - **smoothness term** $S(u,v)$ penalizes dev. from smoothness of the solution
  - regularization parameter $\alpha > 0$ determines the degree of smoothness

- *Remarks:* The minimising functions u and v
  - fit best to all model assumptions (smallest value for the energy functional)
  - can be seen as a **compromise** between all (partly contradictive) assumptions

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The Method of Horn and Schunck

- *Idea:* Assume overall smoothness of the resulting flow field
- The method of Horn and Schunck computes the optical flow as minimizer of *(Horn/Schunck AI 1981)*

$$E(\mathbf{w}) = \int_\Omega \underbrace{(I_x u + I_y v + I_t)^2}_{\text{data term}} + \alpha \underbrace{\left(|\nabla u|^2 + |\nabla v|^2\right)}_{\text{smoothness term}} \, dx\, dy \ .$$

  - **data term** penalizes deviations from the linearized brightness constancy assumption (BCCE)
  - **smoothness term** penalizes deviations from smoothness of the flow field, i.e. from variations of the functions u and v given by their first derivatives

Why variational methods can compute a solution everywhere?

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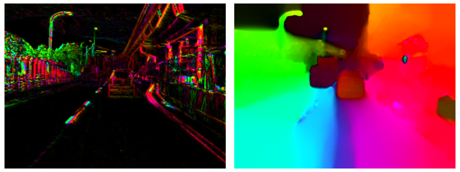
The Filling-In-Effect

- *Observation:* If no information is available, i.e. $|\nabla f| \approx 0$, the flow functions u and v have hardly any influence on the contribution of the data term

$$(f_x u + f_y v + f_t)^2 \approx f_t^2 \ .$$

- *Consequence:* The flow functions u and v adapt to the local solution(s) of the neighborhood to fulfill at least the smoothness term $\rightarrow$ **filling-in-effect**.

edge information  filling-in

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The Motion Tensor Notation

- *Idea:* Rewrite a linearized quadratic data term in a more compact way *(e.g. Bigün et al. TPAMI 1991, Farnebäck ICCV 2001, Bruhn et al. IJCV 2005)*
- *Example:* Linearized gray value constancy assumption (BCCE)

$$(I_x u + I_y v + I_t)^2 = (\mathbf{w}^\top \nabla_3 I)^2 = \mathbf{w}^\top \nabla_3 I \ \nabla_3 I^\top \mathbf{w} = \mathbf{w}^\top J \ \mathbf{w}$$

yields a single quadratic form with the 3×3 **motion tensor**

$$J = \begin{pmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{23} & J_{33} \end{pmatrix} = \begin{pmatrix} I_x^2 & I_x I_y & I_x I_t \\ I_x I_y & I_y^2 & I_y I_t \\ I_x I_t & I_y I_t & I_t^2 \end{pmatrix} = \nabla_3 I \ \nabla_3 I^\top \ .$$

- *Application:* In motion tensor notation the Horn and Schunck method reads

$$E(\mathbf{w}) = \int_\Omega \underbrace{\mathbf{w}^\top J \ \mathbf{w}}_{\text{data term}} + \alpha \underbrace{(|\nabla u|^2 + |\nabla v|^2)}_{\text{smoothness term}} \ dx \, dy \ .$$

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Minimization of Continuous Energy Functionals

- *Idea:* Similar strategy as for ordinary functions $\rightarrow$ derive necessary conditions
- These necessary conditions are called **Euler-Lagrange equations**. They state that the first variation of the energy functional must vanish ($\approx$ first derivative). *(e.g. Elsgolc 1961, Gelfand/Fomin 2000)*
- For a typical optical flow energy functional of type

$$E(u,v) = \int_\Omega F(x, y, u, v, u_x, u_y, v_x, v_y) \, dx \, dy$$

the Euler-Lagrange equations are given by the following system of PDEs

$$0 \stackrel{!}{=} F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} \ ,$$

$$0 \stackrel{!}{=} F_v - \frac{\partial}{\partial x} F_{v_x} - \frac{\partial}{\partial y} F_{v_y}$$

with the associated **boundary conditions** $\mathbf{n}^\top \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} = 0$ and $\mathbf{n}^\top \begin{pmatrix} F_{v_x} \\ F_{v_y} \end{pmatrix} = 0$.

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How Do These Equations Look Like for the Method of Horn and Schunck?

- For the Method of Horn and Schunck $F(x, y, u, v, u_x, u_y, v_x, v_y)$ is given by

$$\begin{aligned} F &= \mathbf{w}^\top J \ \mathbf{w} + \alpha \, (|\nabla u|^2 + |\nabla v|^2) \\ &= J_{11} u^2 + J_{22} v^2 + J_{33} + 2 J_{12} u v + 2 J_{13} u + 2 J_{23} v + \alpha \, (u_x^2 + u_y^2 + v_x^2 + v_y^2) \ . \end{aligned}$$

- The required partial derivatives can then be computed as

$$F_u = 2J_{11} u + 2J_{12} v + 2J_{13} \ , \quad F_{u_x} = \alpha \ 2u_x \ , \quad F_{u_y} = \alpha \ 2u_y \ ,$$
$$F_v = 2J_{12} u + 2J_{22} v + 2J_{23} \ , \quad F_{v_x} = \alpha \ 2v_x \ , \quad F_{v_y} = \alpha \ 2v_y \ .$$

- As necessary condition for a minimizer this yields the Euler–Lagrange equations

$$\begin{aligned} 0 &= F_u - \tfrac{\partial}{\partial x} F_{u_x} - \tfrac{\partial}{\partial y} F_{u_y} = \cancel{2} \big(J_{11} u + J_{12} v + J_{13} - \alpha \, \overbrace{(u_{xx} + u_{yy})}^{\Delta u} \big) \\ 0 &= F_v - \tfrac{\partial}{\partial x} F_{v_x} - \tfrac{\partial}{\partial y} F_{v_y} = \cancel{2} \big(J_{12} u + J_{22} v + J_{23} - \alpha \, \underbrace{(v_{xx} + v_{yy})}_{\Delta v} \big) \end{aligned}$$

with (reflecting) **Neumann boundary conditions** $\mathbf{n}^\top \nabla u = 0$ and $\mathbf{n}^\top \nabla v = 0$.

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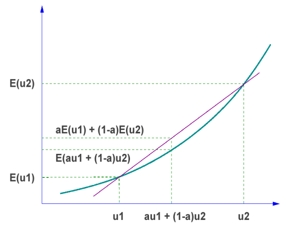
Existence and Uniqueness of the Minimizer

- Strictly convex energy functionals
  - fulfill for all $\alpha \in [0, ..., 1]$ the inequality:

    $$E(\alpha\mathbf{u}_1+(1-\alpha)\mathbf{u}_2) < \alpha E(\mathbf{u}_1)+(1-\alpha)E(\mathbf{u}_2)\,.$$

  - have at most one solution which is **unique** if it exists (global minimizer)



- Further properties of strictly convex variational optical flow methods
  (Schnörr JMIV 1994, Weickert/Schnörr IJCV 2001)
  - existence of a solution
  - solution depends continuously on the input data

Well-posedness (in the sense of Hadamard)

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How Can We Solve The Euler-Lagrange-Equations Numerically?

- *Idea:* Discretize the Euler-Lagrange equations of the Horn and Schunck method

$$\begin{aligned} 0 &= J_{11}u + J_{12}v + J_{13} - \alpha\Delta u \\ 0 &= J_{12}u + J_{22}v + J_{23} - \alpha\Delta v \end{aligned}$$

on a rectangular grid with spacing h_x in x-direction and spacing h_y in y-direction.

- *Solution:* Approximate occurring derivatives via finite differences
  (e.g. Sobel, Scharr, Prewitt, Kumar operators)
  - image derivatives f_x, f_y, f_t required for motion tensor entries J_{nm}
  - flow derivatives $\Delta = u_{xx} + u_{yy}$, $\Delta v = v_{xx} + v_{yy}$, here discretized via

$$\Delta u = \frac{u_{i+1,j} - u_{i,j}}{h_x^2} + \frac{u_{i-1,j} - u_{i,j}}{h_x^2} + \frac{u_{i,j+1} - u_{i,j}}{h_y^2} + \frac{u_{i,j-1} - u_{i,j}}{h_y^2}\,.$$

- *Consistency:* for $h_x \to 0$ and $h_y \to 0$ one obtains the continuous derivatives

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Discrete Euler-Lagrange Equations

- The discrete Euler-Lagrange equations for the method of Horn and Schunck can finally be written as

$$\begin{aligned} 0 &= [J_{11}]_{i,j}\, u_{i,j} + [J_{12}]_{i,j}\, v_{i,j} + [J_{13}]_{i,j} - \alpha \sum_{l\in x,y} \sum_{(\tilde{i},\tilde{j})\in\mathcal{N}_l(i,j)} \frac{u_{\tilde{i},\tilde{j}} - u_{i,j}}{h_l^2} \\ 0 &= [J_{12}]_{i,j}\, u_{i,j} + [J_{22}]_{i,j}\, v_{i,j} + [J_{23}]_{i,j} - \alpha \sum_{l\in x,y} \sum_{(\tilde{i},\tilde{j})\in\mathcal{N}_l(i,j)} \frac{v_{\tilde{i},\tilde{j}} - v_{i,j}}{h_l^2} \end{aligned}$$

for $i = 1, ..., N$ and $j = 1, ..., M$.

  - here, $\mathcal{N}_l(i,j)$ denotes the set of neighbors of pixel i, j in direction of axis l (assuming four direct neighbors, i.e. two in each direction)
  - these equations constitute a **linear system of equations** w.r.t. the $2N \times M$ unknowns $u_{i,j}$ and $v_{i,j}$ for $i = 1, ..., N$ and $j = 1, ..., M$

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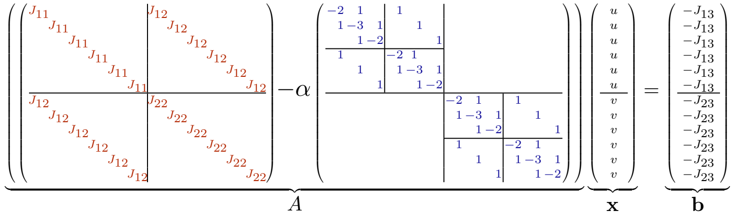
Structure of the Linear System

- This linear system of equations $A\mathbf{x} = \mathbf{b}$ has the following block structure



  - **smoothness term** only contributes to block main diagonals
  - **data term** also contributes to block off-diagonals

- For non-constant input images the matrix A is positive definite
- For an image with 1M pixels, the matrix A has $4 \cdot 10^{12}$ entries. Assuming 32-bit float precision this requires 16 Terabyte memory ($\rightarrow$ store only non-zero entries). Direct Gauss-Elimination with complexity $O(n^3)$ is not practicable.

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Solving Linear Systems of Equations

How Can We Solve The Linear System of Equation $A\mathbf{x} = \mathbf{b}$?

- *Idea:* Find a cheap but accurate approximation of A^{-1} via the decomposition *(e.g. Young 1971, Saad 1996)*

$$A = A_1 + A_2$$

- Introduce fixed point iteration of type

$$A_1\, \mathbf{x}^{k+1} = \mathbf{b} - A_2\, \mathbf{x}^{k}$$

$$\Leftrightarrow \mathbf{x}^{k+1} = A_1^{-1}(\mathbf{b} - A_2\, \mathbf{x}^{k})$$

- In each iteration a linear system of equations with matrix A_1 has to be solved
  - A_1^{-1} should be a reasonable approximation of A^{-1}
  - A_1^{-1} should be cheap to compute, i.e. the system should be simple to solve

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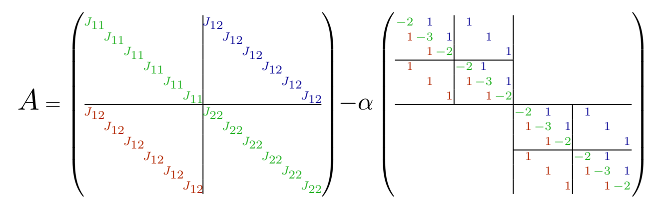
Frequent Approach

- Use matrix decomposition of type

$$A = D - L - U\,.$$

  - D is the diagonal part of A
  - L is the strictly lower triangular part of A
  - U is the strictly upper triangular part of A

- For the method of Horn and Schunck this yields



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Frequent Approach

- In terms of the discretized Euler-Lagrange equations we obtain

$$\begin{aligned} 0 &= [J_{11}]_{i,j}\, u_{i,j} + [J_{12}]_{i,j}\, v_{i,j} + [J_{13}]_{i,j} \\ &\quad + \alpha \sum_{l\in x,y} \sum_{(\tilde i,\tilde j)\in\mathcal{N}_l(i,j)} \frac{1}{h_l^2}\, u_{i,j} - \alpha \sum_{l\in x,y} \sum_{(\tilde i,\tilde j)\in\mathcal{N}_l^-(i,j)} \frac{1}{h_l^2}\, u_{\tilde i,\tilde j} - \alpha \sum_{l\in x,y} \sum_{(\tilde i,\tilde j)\in\mathcal{N}_l^+(i,j)} \frac{1}{h_l^2}\, u_{\tilde i,\tilde j} \\ 0 &= [J_{12}]_{i,j}\, u_{i,j} + [J_{22}]_{i,j}\, v_{i,j} + [J_{23}]_{i,j} \\ &\quad + \alpha \sum_{l\in x,y} \sum_{(\tilde i,\tilde j)\in\mathcal{N}_l(i,j)} \frac{1}{h_l^2}\, v_{i,j} - \alpha \sum_{l\in x,y} \sum_{(\tilde i,\tilde j)\in\mathcal{N}_l^-(i,j)} \frac{1}{h_l^2}\, v_{\tilde i,\tilde j} - \alpha \sum_{l\in x,y} \sum_{(\tilde i,\tilde j)\in\mathcal{N}_l^+(i,j)} \frac{1}{h_l^2}\, v_{\tilde i,\tilde j} \end{aligned}$$

for $i = 1, ..., N$ and $j = 1, ..., M$.

- Notation for the neighborhood
  - $\mathcal{N}_l^-(i,j)$ denotes the set of neighbors of pixel i,j in direction of axis l that have a **smaller index** (will be updated **before** the central pixel)
  - $\mathcal{N}_l^+(i,j)$ denotes the set of neighbors of pixel i,j in direction of axis l that have a **larger index** (will be updated **after** the central pixel)

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The Jacobi Method

- Set $A_1 = D$, since diagonal matrices are simple to invert. $A_2 = -L - U$.
- Yields the fixed point iteration

$$\mathbf{x}^{k+1} = D^{-1}(\,\mathbf{b} + (L+U)\,\mathbf{x}^{k}\,) \quad\Leftrightarrow\quad x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij} x_j^k - \sum_{j>i} a_{ij} x_j^k\right).$$

- For the Horn and Schunck method the Jacobi iteration for the pixel i,j reads

$$u_{i,j}^{k+1} = \frac{\left(-[J_{13}]_{i,j} - \left([J_{12}]_{i,j}\, v_{i,j}^{k} - \alpha \sum\limits_{l\in x,y} \sum\limits_{\mathcal{N}_l^-(i,j)} \frac{1}{h_l^2}\, u_{\tilde i,\tilde j}^{k} - \alpha \sum\limits_{l\in x,y} \sum\limits_{\mathcal{N}_l^+(i,j)} \frac{1}{h_l^2}\, u_{\tilde i,\tilde j}^{k}\right)\right)}{[J_{11}]_{i,j} + \alpha \sum\limits_{l\in x,y} \sum\limits_{(\tilde i,\tilde j)\in\mathcal{N}_l(i,j)} \frac{1}{h_l^2}},$$

$$v_{i,j}^{k+1} = \frac{\left(-[J_{23}]_{i,j} - \left([J_{12}]_{i,j}\, u_{i,j}^{k} - \alpha \sum\limits_{l\in x,y} \sum\limits_{\mathcal{N}_l^-(i,j)} \frac{1}{h_l^2}\, v_{\tilde i,\tilde j}^{k} - \alpha \sum\limits_{l\in x,y} \sum\limits_{\mathcal{N}_l^+(i,j)} \frac{1}{h_l^2}\, v_{\tilde i,\tilde j}^{k}\right)\right)}{[J_{22}]_{i,j} + \alpha \sum\limits_{l\in x,y} \sum\limits_{(\tilde i,\tilde j)\in\mathcal{N}_l(i,j)} \frac{1}{h_l^2}}.$$

Iterative Solvers (5)

The Gauß-Seidel Method

- Set $A_1 = D - L$, since this triangular matrix is a better approximation to A than the diagonal D alone. Triangular matrices are still simple to invert. $A_2 = -U$.
- Yields the fixed point iteration

$$\mathbf{x}^{k+1} = (D-L)^{-1}(\mathbf{b} + U\,\mathbf{x}^k) \quad \Leftrightarrow \quad x_i^{k+1} = \frac{1}{a_{ii}}\left(b_i - \sum_{j<i} a_{ij} x_j^{\boxed{k+1}} - \sum_{j>i} a_{ij} x_j^k\right).$$

- The corresponding Gauß-Seidel iteration for the pixel i,j reads

$$u_{i,j}^{k+1} = \frac{\left(-[J_{13}]_{i,j} - \left([J_{12}]_{i,j}\, v_{i,j}^{k} - \alpha \sum_{l \in x,y} \sum_{\mathcal{N}_l^-(i,j)} \frac{1}{h_l^2} u_{\tilde{i},\tilde{j}}^{\boxed{k+1}} - \alpha \sum_{l \in x,y} \sum_{\mathcal{N}_l^+(i,j)} \frac{1}{h_l^2} u_{\tilde{i},\tilde{j}}^{k}\right)\right)}{[J_{11}]_{i,j} + \alpha \sum_{l \in x,y} \sum_{(\tilde{i},\tilde{j}) \in \mathcal{N}_l(i,j)} \frac{1}{h_l^2}},$$

$$v_{i,j}^{k+1} = \frac{\left(-[J_{23}]_{i,j} - \left([J_{12}]_{i,j}\, u_{i,j}^{\boxed{k+1}} - \alpha \sum_{l \in x,y} \sum_{\mathcal{N}_l^-(i,j)} \frac{1}{h_l^2} v_{\tilde{i},\tilde{j}}^{\boxed{k+1}} - \alpha \sum_{l \in x,y} \sum_{\mathcal{N}_l^+(i,j)} \frac{1}{h_l^2} v_{\tilde{i},\tilde{j}}^{k}\right)\right)}{[J_{22}]_{i,j} + \alpha \sum_{l \in x,y} \sum_{(\tilde{i},\tilde{j}) \in \mathcal{N}_l(i,j)} \frac{1}{h_l^2}}.$$

Iterative Solvers (6)

Remarks to the Gauß-Seidel Method

- Advantages
  - positive definiteness of the matrix A sufficient for convergence
  - about twice as fast as the Jacobi technique
  - does not require to store values from the previous iteration k (less memory consumption, easier to implement)
- Drawbacks
  - more difficult to parallelize than the Jacobi method (see PART III)
  - performance depends on the order in which the unknowns are traversed (symmetric variants exist that partly account for that problem)
  - still far from being real-time capable for small images sizes
- Outlook
  - in PART III we will discuss much more advanced numerical schemes based on the Gauß-Seidel method that even allow for real-time performance

Results (1)

Results

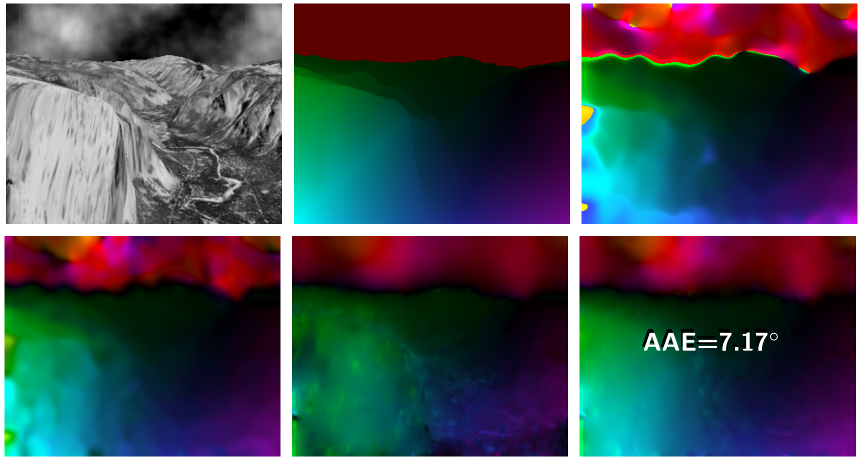
Comparison to Classical Approaches w.r.t. the Average Angular Error (AAE)

- Qualitative Evaluation for the Yosemite Sequence with Clouds

Technique	AAE
Normal Flow	55.56°
Normalized Cross Correlation (NCC)	21.84°
Block Matching + Subpixel (SSD)	21.46°
Horn and Schunck (2-D)	**13.29°**
Bigün et al. + Presmoothing (2-D)	10.60°
Lucas/Kanade + Presmoothing (2-D)	8.79°
Horn and Schunck + Presmoothing (2-D)	**7.17°**

Results (2)

Results for the Horn and Schunck Method



Results for the Yosemite Sequence with clouds (L. Quam). **(a) Upper Left:** Frame 8. **(b) Upper Center:** Ground truth. **(c) Upper Right:** Bigün et al. **(d) Lower Left:** Lucas/Kanade. **(d) Lower Center:** Horn and Schunck w/o presmoothing. **(d) Lower Right:** Horn and Schunck with presmoothing.

Summary

- Variational methods compute optical flow as minimizer of an energy functional
- They make use of global smoothness assumptions on the solution to overcome the aperture problem (filling-in-effect by the smoothness term $\rightarrow$ dense results)
- They are minimized by solving their (discretized) Euler-Lagrange equations
- They offer many advantages such as
  - transparent modeling
  - dense flow fields
  - well-posedness
  - sub-pixel precision
- The method of Horn and Schunck is the simplest variational approach
- There are many adaptations/modifications of this basic method possible that improve the quality and the performance even further (see PART II-III)