Numerical Algorithms for Visual Computing 2010/11
Example Solutions for Assignment 2

Problem 1 (Different Differentials)

Our first task is to compute the Jacobian matrix J. We start with the partial derivatives in x which form the first column vector of J:

$$\begin{aligned}
\frac{\partial S_1}{\partial x} &= \frac{\partial}{\partial x}\frac{fu(x,y)x}{\sqrt{x^2+y^2+f^2}} \\
&= \frac{f(u_x x+u)\sqrt{x^2+y^2+f^2}}{x^2+y^2+z^2} - \frac{fux^2}{(x^2+y^2+f^2)^{\frac{3}{2}}} \\
&= f\frac{u_x x^3+u_x xy^2+u_x xf^2+uy^2+uf^2}{(x^2+y^2+f^2)^{\frac{3}{2}}} \\
\frac{\partial S_2}{\partial x} &= \frac{\partial}{\partial x}\frac{fu(x,y)y}{\sqrt{x^2+y^2+f^2}} \\
&= \frac{f(u_x y)\sqrt{x^2+y^2+f^2}}{x^2+y^2+f^2} - \frac{fuxy}{(x^2+y^2+f^2)^{\frac{3}{2}}} \\
&= f\frac{u_x x^2 y+u_x y^3+u_x yf^2-uxy}{(x^2+y^2+f^2)^{\frac{3}{2}}} \\
\frac{\partial S_3}{\partial x} &= \frac{\partial}{\partial x}\frac{f^2u(x,y)}{\sqrt{x^2+y^2+f^2}} \\
&= \frac{f^2u_x\sqrt{x^2+y^2+f^2}}{x^2+y^2+f^2} - \frac{f^2ux}{(x^2+y^2+f^2)^{\frac{3}{2}}} \\
&= f^2\frac{u_x x^2+u_x y^2+u_x f^2-ux}{(x^2+y^2+f^2)^{\frac{3}{2}}}
\end{aligned}$$

In the same way, we get the corresponding partial derivatives in y:

$$\begin{aligned}
\frac{\partial S_1}{\partial y} &= f\frac{u_y x^3+u_y xy^2+u_y xf^2-uxy}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
\frac{\partial S_2}{\partial y} &= f\frac{u_y x^2 y+u_y y^3+u_y yf^2+ux^2+uf^2}{(x^2+y^2+f^2)^{\frac{3}{2}}} \\
\frac{\partial S_3}{\partial y} &= f^2\frac{u_y x^2+u_y y^2+u_y f^2-uy}{(x^2+y^2+f^2)^{\frac{3}{2}}}
\end{aligned}$$

Now, we need to verify:

$$n(x,y) = \begin{pmatrix} \frac{\partial S_1}{\partial x} \\ \frac{\partial S_2}{\partial x} \\ \frac{\partial S_3}{\partial x} \end{pmatrix} \times \begin{pmatrix} \frac{\partial S_1}{\partial y} \\ \frac{\partial S_2}{\partial y} \\ \frac{\partial S_3}{\partial y} \end{pmatrix}$$

In order to improve readability we define $A := x^2 + y^2 + f^2$.

$$\begin{aligned}
\frac{\partial S_3}{\partial x}\frac{\partial S_1}{\partial y} - \frac{\partial S_1}{\partial x}\frac{\partial S_3}{\partial y} &= f\frac{u_y x(x^2+y^2+f^2) - uxy}{(x^2+y^2+f^2)^{\frac{3}{2}}} f^2 \frac{u_x(x^2+y^2+f^2) - ux}{(x^2+y^2+z^2)^{\frac{3}{2}}} \\
&- f\frac{u_x x(x^2+y^2+f^2) + u(y^2+f^2)}{(x^2+y^2+f^2)^{\frac{3}{2}}} f^2 \frac{u_y A - uy}{(x^2+y^2+f^2)^{\frac{3}{2}}} \\
&= f^3 \frac{u_x u_y x^2 A^2 - u u_y x^2 A - u u_x xyA + u^2 x^2 y}{(x^2+y^2+f^2)^{\frac{3}{2}}} \\
&- f^3 \frac{u_x u_y x^2 A^2 - u u_x xyA + u u_y A(y^2+f^2) - u^2 y(y^2+f^2)}{(x^2+y^2+f^2)^{\frac{3}{2}}} \\
&= f^3 \frac{u^2 yA - u u_y A^2}{A^3} = f^3 \frac{u^2 y - u u_y A}{A^2} \\
&= -(\frac{f^2 u}{A})(f u_y - \frac{fu}{x^2+y^2+f^2}) = -(\frac{f^2 u}{A})\mathbf{n}(x,y)_2
\end{aligned}$$

In a very similar way we also show that $\frac{\partial S_2}{\partial x}\frac{\partial S_3}{\partial y} - \frac{\partial S_3}{\partial x}\frac{\partial S_2}{\partial y} = -(\frac{f^2 u}{A})\mathbf{n}(x,y)_1$. Thus, only one final computation remains:

$$\begin{aligned}
\frac{\partial S_1}{\partial x}\frac{\partial S_2}{\partial y} - \frac{\partial S_2}{\partial x}\frac{\partial S_1}{\partial y} &= f\frac{u_x xA + u(y^2+f^2)}{A^{\frac{3}{2}}} f \frac{u_y yA + u(x^2+f^2)}{A^{\frac{3}{2}}} \\
&- f\frac{u_x yA - uxy}{A^{\frac{3}{2}}} f \frac{u_y xA - uxy}{A^{\frac{3}{2}}} \\
&= f^2 \frac{u_x u_y xyA^2 + u u_x xA(x^2+f^2) + u u_y yA(y^2+f^2)}{A^3} \\
&+ f^2 \frac{u^2(x^2+f^2)(y^2+f^2)}{A^3} \\
&- f^2 \frac{u_x u_y xyA^2 - u u_x xy^2 A - u u_y x^2 yA + u^2 x^2 y^2}{A^3} \\
&= f^2 \frac{u u_x xA^2 u + u_y yA^2 + u^2(x^2y^2 + (x^2+y^2)f^2 + f^4 - x^2y^2)}{A^3} \\
&= \frac{f^2 u}{A}\left(\nabla u \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{f^2 u}{A}\right) = \frac{f^2 u}{A}\mathbf{n}(x,y)_3
\end{aligned}$$

Problem 2 (Potentially Elliptic)

Let us first compute the derivatives of $u(x, y, z)$.

$$\begin{aligned}
\frac{\partial}{\partial x} u(x,y,z) &= \frac{8\pi a^3 x}{3(x^2+y^2+z^2)^{\frac{3}{2}}} \\
\frac{\partial^2}{\partial^2 x} u(x,y,z) &= \frac{8\pi a^3((x^2+y^2+z^2)^{\frac{3}{2}} - 3x^2(x^2+y^2+z^2)^{\frac{1}{2}})}{3(x^2+y^2+z^2)^{\frac{3}{2}}} \\
&= \frac{8\pi a^3(x^2+y^2+z^2-3x^2)}{x^2+y^2+z^2} = \frac{8\pi a^3(-2x^2+y^2+z^2)}{x^2+y^2+z^2}
\end{aligned}$$

The derivatives in y and z are virtually identical except for a swapping of variables in the numerator of the respective fractions.

Consider now that $(\cdot)^q$ with $q \in \mathbb{Q}$ are continuous functions on $\mathbb{R}$. Thus, u and its derivatives are combinations of continuous functions and thereby continuous on their whole domain.

By plugging the second derivatives of u into the Laplace equation we instantly see that the claim of part b) is true:

$$\frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} + \frac{\partial^2 u}{\partial^2 z} = \frac{8\pi a^3(3(x^2+y^2+z^2) - 3x^2 - 3y^2 - 3z^2))}{x^2+y^2+z^2} = 0$$

Problem 3 (Law and Orders)

We compute the local truncation error of A in respect to $u'((i - 1\frac{1}{2})\delta x)$ in the same way as in the lecture notes. First, we associate

$$\begin{cases} x &= i - \frac{1}{2}\Delta x \\ x - \frac{1}{2}\Delta x &= i - \Delta x \\ x + \frac{1}{2}\Delta x &= i\Delta x \end{cases}$$

Thus, we can write

$$A = \frac{u(x+\frac{1}{2}\Delta x) - u(x - \frac{1}{2}\Delta x)}{\Delta x} \tag{1}$$

In order to compute the local truncation error we need the Taylor expansions of the mesh points that are used in the approximation:

$$\begin{aligned}
u(x+\frac{1}{2}\Delta x) &= u + \frac{\Delta x}{2}u' + \frac{\Delta x^2}{8}u'' + \frac{\Delta x^3}{16} + \mathcal{O}(\Delta x^4) \\
u(x+\frac{1}{2}\Delta x) &= u - \frac{\Delta x}{2}u' + \frac{\Delta x^2}{8}u'' - \frac{\Delta x^3}{16} + \mathcal{O}(\Delta x^4)
\end{aligned}$$

Now we can compute the local truncation error by plugging the Taylor expansions into equation 1:

$$\begin{aligned} L_{\Delta x}(u(x)) &= \frac{u + \frac{\Delta x}{2}u' + \frac{\Delta x^2}{8}u'' + \frac{\Delta x^3}{16}u''' + \mathcal{O}(\Delta x^4)}{\Delta x} \\ &- \frac{u - \frac{\Delta x}{2}u' + \frac{\Delta x^2}{8}u'' - \frac{\Delta x^3}{16}u''' + \mathcal{O}(\Delta x^4)}{\Delta x} \\ &= u' + \frac{\Delta x^2}{8}u''' + \mathcal{O}(\Delta x^4) = u' + \mathcal{O}(\Delta x^2) \end{aligned}$$
