Numerical Algorithms for Visual Computing 2010/11
Example Solutions for Assignment 1

Problem 1 (Typesetting of PDEs) First of all, let us have a closer look at what we are doing here. In general, the differential equation that we see, can be described in terms of a conic section. For more information, please refer to the mathematics for computer science lectures on quadrics. One can see a differential equation in terms of a quadric by computing the following

$$(x, y, 1) \begin{pmatrix} a & b & d \\ b & c & e \\ d & e & f \end{pmatrix} (x, y, 1)^\top = g$$

which happens to be, after some calculation, the equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = g,$$

and for sake of simplicity, the factors in front of the b,d,e are omitted (they can be incorporated into the corresponding coefficients). However, we are mostly interested in the upper 2×2-matrix and its resulting quadric

$$(x, 1) \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix} (x, 1)^\top = ax^2 + bx + c.$$

If we now have a look at the discriminant

$$\begin{vmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{vmatrix} = ac - \frac{1}{4}b^2 \stackrel{!}{=} 0,$$

we can see that this our sought relation. Furthermore, if we have a look back at the equation $ax^2 + bx + c$, this equation is quite well known from school, it has two solutions

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

and, depending on the sign of the square root, has 0, 1 or 2 solutions. The equation $b^2 - 4ac > 0$ gives two solutions, equal zero has only one solution if it is negative, it has no real solution. Let us come back to our problem, this gives an analogon for the categorisation of the differential equation. In case, the discriminant is equal to zero, we have a parabolic equation, in case we have no solution, we have an elliptic equation and for two solutions, we have a hyperbolic equation.

1. $u_t = u_{xx}$ (Diffusion equation)
   We have $A = 1$, $B = C = D = F = G = 0$ and $E = -1$, therefore, $B^2 - 4AC = 0$.
   Parabolic, Second order, linear

2. $u_{tt} = u_{xx}$ (Wave equation)
   $A = 1$, $C = -1$, $B = D = E = F = G = 0$, i.e. $B^2 - 4AC = 4 > 0$.
   Hyperbolic, Second order, linear

3. $u_{xx} + u_{yy} = 0$ (Laplace equation)
   $A = C = 1$, $B = D = E = F = G = 0$, i.e. $B^2 - 4AC = -4 < 0$.
   Elliptic, Second order, linear

4. $xu_x + yu_y + u^2 = 0$
   $A = B = C = 0$,$D = x$,$E = y$,$F = u$, i.e. $B^2 - 4AC = 0$.
   Parabolic, First order, non-linear. The non-linearity comes from the coefficient $F = u$ which yields the nonlinear term $Fu = u^2$. Note that the terms x and y are so-called variable coefficients and, in general, those terms do not imply non-linearity.

Problem 2 (Tayloring schemes) While the task in problem 2 was to use Taylor expansions in order to approximate the second derivative, we first recall the finite difference method from the lecture as an alternative computation method.

<u>Method 1:</u>
Analogously to the scheme (2.6) in the script, we compute $u''(j\Delta x)$ using the central difference scheme. The building blocks of the second derivative approximation are the following two first derivative approximations:

$$\begin{aligned} u'((j+1)\Delta x) &\approx \frac{u_{j+2} - u_j}{2\Delta x} \\ u'((j-1)\Delta x) &\approx \frac{u_j - u_{j-2}}{2\Delta x} \end{aligned}$$

Those can be combined to the sought second derivative approximation:

$$\begin{aligned} u''(j\Delta x) &\approx \frac{u''((j+1)\Delta x) - u''((j-1)\Delta x)}{2\Delta x} \\ &\approx \frac{\frac{u_{j+2} - u_j}{2\Delta x} - \frac{u_j - u_{j-2}}{2\Delta x}}{2\Delta x} \\ &= \frac{u_{j-2} - 2u_j + u_{j+2}}{4\Delta x^2} \end{aligned}$$

Method 2: Taylor-Expansion

$$
\begin{aligned}
u(j+2\Delta x) = u_{j+2} &= u_j + 2\Delta x u_j' + 2\Delta x^2 u_j'' + \frac{4}{3}\Delta x^3 u_j''' + \frac{2}{3}\Delta x^4 u_j'''' + \mathcal{O}(\Delta x^5) \\
u_j &= u_j \\
u(j-2\Delta x) = u_{j-2} &= u_j - 2\Delta x u_j' + 2\Delta x^2 u_j'' - \frac{4}{3}\Delta x^3 u_j''' + \frac{2}{3}\Delta x^4 u_j'''' + \mathcal{O}(\Delta x^5)
\end{aligned}
$$

Now, in order to compute the sought derivative scheme, we need two additional equations for our scheme.

Our approximation of the second derivative should use only the mesh points u_{j-2}, u_j and u_{j+2}, thus yielding the ansatz:

$$
u_j'' = \alpha u_{j-2} + \beta u_j + \gamma u_{j+2} \tag{1}
$$

Additionally, we know that:

$$
u_j'' = 0 \cdot u_j + 0 \cdot u_j' + 1 \cdot u_j'' \tag{2}
$$

Plugging the Taylor expansions into equation 1 will give us a representation of u_j'' in terms of u_j and its derivatives. First, we discard the higher order derivatives and only focus on the terms with order one to two:

$$
\begin{aligned}
& u_j \underbrace{(\alpha + \beta + \gamma)}_{=0} \\
+ \quad & u_j' \Delta x \underbrace{(-2\alpha + 2\gamma)}_{=0} \\
+ \quad & u_j'' \Delta x^2 \underbrace{(2\alpha + 2\gamma)}_{=1}
\end{aligned}
$$

Comparing the coefficients of the derivatives of the newly acquired equation 3 and with the corresponding coefficients of equation 2 allows us to compute α, β, γ via a linear system of equations:

$$
\begin{pmatrix} 1 & 1 & 1 \\ -2 & 0 & 2 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\Delta x^2} \end{pmatrix}
$$

Solving the system yields $\alpha = \gamma = \frac{1}{4\Delta x^2}$ and $\beta = -\frac{1}{2\Delta x^2}$, which coincides with the coefficients computed in method 1. In order to compute the local truncation error, we also need the Δx^3 and Δx^4 terms that we ignored in equation 2:

$$\begin{aligned}
& u_j(\alpha+\beta+\gamma) \\
+ \quad & u_j'\Delta x(-2\alpha+2\gamma) \\
+ \quad & u_j''\Delta x^2(2\alpha+2\gamma) \\
+ \quad & u_j'''\Delta x^3(\frac{4}{3}\gamma-\frac{4}{3}\alpha) \\
+ \quad & u_j''''\Delta x^4(\frac{2}{3}\alpha+\frac{2}{3}\gamma)+\mathcal{O}(\Delta x^5)
\end{aligned}$$

Now for the local truncation error, we compute the first non-vanishing term with a higher order than the approximated derivative u''. The Δx^3-term vanishes, whereas the fourth order derivative does not vanish, i.e.

$$L_{\Delta x}(u) = u''+\frac{1}{3}u_j''''\Delta x^2+\mathcal{O}(\Delta x^5) = u''+\mathcal{O}(\Delta x^2) = \mathcal{O}(\Delta x^2)$$

Problem 3 (How big is this $\mathcal{O}$?) Remember:

$$\varphi(h)=\mathcal{O}(h^p) \quad \Leftrightarrow \quad \lim_{h\to 0}\frac{\varphi(h)}{h^p}=C<\infty.$$

1. $\mathcal{O}(h^p)+\mathcal{O}(h^q)=\mathcal{O}(h^p)$
   Let $\varphi_1(h)=\mathcal{O}(h^p)$ and $\varphi_2(h)=\mathcal{O}(h^q)$. This can be introduced into the definition

$$\begin{aligned}
\lim_{h\to 0}\frac{\varphi_1(h)+\varphi_2(h)}{h^p} &= \lim_{h\to 0}\frac{\varphi_1(h)}{h^p}+\frac{\varphi_2(h)}{h^p} \\
&= \lim_{h\to 0}\frac{\varphi_1(h)}{h^p}+\lim_{h\to 0}\frac{\varphi_2(h)}{h^p} \\
&= \lim_{h\to 0}\frac{\varphi_1(h)}{h^p}+\lim_{h\to 0}\frac{\varphi_2(h)}{h^q\cdot h^{p-q}} \\
&= \underbrace{\lim_{h\to 0}\frac{\varphi_1(h)}{h^p}}_{\mathcal{O}(h^p)}+\underbrace{\lim_{h\to 0}\frac{\varphi_2(h)}{h^q}}_{\mathcal{O}(h^q)}\cdot\underbrace{\lim_{h\to 0}\frac{1}{h^{p-q}}}_{\to 0} \\
&= |c_1|+\lim_{h\to 0}\frac{1}{h^{p-q}}|c_2|=|c_1| \\
&= \mathcal{O}(h^p)
\end{aligned}$$

2. $\mathcal{O}(h^p) \cdot \mathcal{O}(h^q) = \mathcal{O}(h^{p+q})$

$$\begin{aligned}
\lim_{h\to 0} \frac{\varphi_1(h)\cdot\varphi_2(h)}{h^{p+q}} &= \lim_{h\to 0} \frac{\varphi_1(h)}{h^p}\cdot\frac{\varphi_2(h)}{h^q} \\
&= \lim_{h\to 0} \frac{\varphi_1(h)}{h^p}\cdot\lim_{h\to 0}\frac{\varphi_2(h)}{h^q} \\
&= |c_1|\cdot|c_2| = |c_1\cdot c_2| < \infty
\end{aligned}$$

3. $\mathcal{O}(h^p) - \mathcal{O}(h^p) = \mathcal{O}(h^p)$

$$\begin{aligned}
\lim_{h\to 0} \frac{f(h)-g(h)}{h^p} &= \lim_{h\to 0} \frac{f(h)}{h^p} - \lim_{h\to 0}\frac{g(h)}{h^p} \\
&= \frac{f(h)-g(h)}{h^p} = |c_1| - |c_2| < \infty
\end{aligned}$$

4. As seen in the script, $\frac{1}{\mathcal{O}(h^p)}$ does not exist.

Problem 4 (Resizing $\mathcal{O}$)

$$\begin{aligned}
a(h) &= h + h^2 + 10^{20}h^3 \\
b_1(h) &= h + h^2 + 10^{20}h^3 + 10^{-100}h^4 \\
b_2(h) &= -h - h^2 + 10^{20}h^3 + 10^{-100}h^4.
\end{aligned}$$

1. $a(h)\cdot b(h)$

$$\begin{aligned}
a(h)\cdot b(h) &= (h + h^2 + 10^{20}h^3)\cdot(h + h^2 + 10^{20}h^3 + 10^{-100}h^4) \\
&= h^2 + h^3 + 10^{20}h^4 + 10^{-100}h^5 + h^3 + h^4 \\
&+ 10^{20}h^5 + 10^{-100}h^6 + 10^{20}h^4 + 10^{20}h^5 + 10^{40}h^6 + 10^{-80}h^7 \\
&= h^2 + 2h^3 + (2\cdot 10^{20}+1)h^4 + (2\cdot 10^{20} + 10^{-100})h^5 \\
&+ (10^{-100} + 10^{40})h^6 + 10^{-80}h^7 \\
&\overset{\text{see 3a}}{=} h^2 + \mathcal{O}(h^2) = \mathcal{O}(h^2)
\end{aligned}$$

2. $b_1(h) - a(h)$

$$\begin{aligned}
b_1(h) - a(h) &= h + h^2 + 10^{20}h^3 + 10^{-100}h^4 - h + h^2 + 10^{20}h^3 \\
&= 10^{-100}h^4 = \mathcal{O}(h^4)
\end{aligned}$$

3. $b_1(h) + b_2(h)$

$$\begin{aligned} b_1(h) + b_2(h) &= h + h^2 + 10^{20}h^3 + 10^{-100}h^4 - h - h^2 + 10^{20}h^3 + 10^{20}h^3 \\ &= 2 \cdot 10^{20}h^3 + 2 \cdot 10^{-100}h^4 \overset{\text{see 3a}}{=} \mathcal{O}(h^3) \end{aligned}$$
