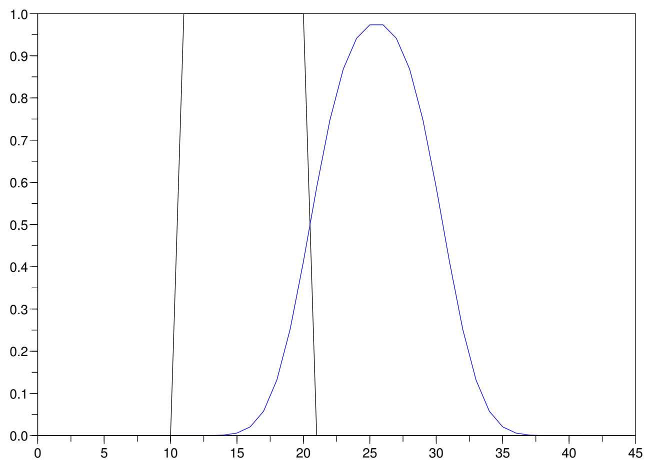
Numerical Algorithms for Visual Computing II 2010/11
Example Solutions for Assignment 7

Problem 1 (Parabolic Stability Reloaded) This exercise is discussed together with its "prequel"-exercise "Parabolic Stability" in the example solution for assignment 6.

Problem 2 (Hyper, Hyper!)

1. We will be using a discretised signal on the interval $[-1, 3]$ with the discretised points having a distance of $\Delta x = 0.1$ between each other, together with the given definition for the signal by equation (3). This gives a standard box function. Concerning boundary conditions, we will use periodic boundary conditions, i.e. $u_0 = u_n$ and $u_{n+1} = u_1$ for the signal with length n. With this setup, the signal propagates towards the right direction, however is then being repeated at the left hand side of the interval. Standard von Neumann boundaries would have the effect of washing the entire signal towards one boundary.



2. **General remarks:**

   - Viscosity solutions satisfy a minimum-maximum-principle enforced by the diffusion term εu_{xx}, see (16.12).
   - Monotone schemes satisfy a discrete minimum-maximum-principle.

Question: Can we recognise a similar form as the viscosity solution provides by analysing the numerical scheme? As an example, we consider the upwind scheme

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) \tag{1}$$

approximating $u_t + au_x = 0, \quad a > 0$. Taylor expansions yield, employing $u \equiv u(j\Delta x, n\Delta t) \approx u_j^n$:

$$u_j^{n+1} = u + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} + \frac{\Delta t^3}{6}u_{ttt} + \mathcal{O}(\Delta t^4) \tag{2a}$$

$$u_j^n = u \tag{2b}$$

$$u_{j-1}^n = u - \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} - \frac{\Delta x^3}{6}u_{xxx} + \mathcal{O}(\Delta x^4) \tag{2c}$$

Computing the local truncation error L, we obtain by use of (2a) and $\lambda = \frac{\Delta t}{\Delta x}$ = constant:

$$\begin{aligned} L &= \frac{[u + \Delta t u_t + \frac{\Delta t^2}{2}u_{tt} + \frac{\Delta t^3}{6}u_{ttt} + \mathcal{O}(\Delta t^4)] - u}{\Delta t} \\ &\quad + \frac{a}{\Delta x}\left[u - \left[u - \Delta x u_x + \frac{\Delta x^2}{2}u_{xx} - \frac{\Delta x^3}{6}u_{xxx} + \mathcal{O}(\Delta x^4)\right]\right] \\ &= \left.\begin{array}{l} u_t + \frac{\Delta t}{2}u_{tt} + \frac{\Delta t^2}{6}u_{ttt} + \mathcal{O}(\Delta t^3) \\ +au_x - \frac{a\Delta x}{2}u_{xx} + \frac{a\Delta x^2}{6}u_{xxx} + \mathcal{O}(\Delta x^3) \end{array}\right\} \qquad (3a) \\ &= \frac{\Delta t}{2}u_{tt} + \frac{\Delta t^2}{6}u_{ttt} - \frac{a\Delta x}{2}u_{xx} + \frac{a\Delta x^2}{6}u_{xxx} + \mathcal{O}(\Delta x^3). \qquad (3b) \end{aligned}$$

We observe, that the upwind scheme is of first order in space and time.

3. In order to give (3a) a further interpretation, we use the approximated PDE $u_t + au_x = 0$ again:

$$\begin{aligned} u_{tt} = (u_t)_t &= (-au_x)_t \\ &= -a(u_t)_x \\ &= -a(-au_x)_x = a^2 u_{xx}. \end{aligned} \tag{4}$$

4. Plugging this into (3a) gives

$$\begin{aligned} L &= \frac{\Delta t}{2}a^2 u_{xx} - \frac{a\Delta x}{2}u_{xx} + \mathcal{O}(\Delta x^2) \\ &= -\frac{a\Delta x}{2}(1 - a\lambda)u_{xx} + \mathcal{O}(\Delta x^2). \end{aligned} \tag{5}$$

Thus, $-\frac{a\Delta x}{2}(1-a\lambda)u_{xx}$ is the leading order error term. The idea is now to subtract the leading order error term from the original equation, thus obtaining a PDE-model for the qualitative behaviour of the scheme.

The statement is here: The upwind scheme

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}(u_j^n - u_{j-1}^n) \tag{6}$$

is a first-order accurate approximation of the PDE

$$u_t + au_x = 0, \tag{7}$$

and a second-order accurate approximation of the modified equation

$$u_t + au_x = \frac{a\Delta x}{2}(1-a\lambda)u_{xx}. \tag{8}$$

5. **Remarks:**

- For $1 - a\lambda \geq 0 \Leftrightarrow \Delta t \leq \frac{\Delta x}{a}$, (8) is of the same form as the advection-diffusion PDE (16.12). Here we observe the link to viscosity solutions, suggesting that the scheme is reasonable.
- The condition $\Delta t \leq \frac{\Delta x}{a}$ is exactly the CFL-condition in the monotonicity analysis. For larger Δt, (8) suggests the influence of backward diffusion, leading to a blow-up of numerical solutions.

To verify the link between (6) and (8), we have to use the same procedure as for the computation of the local truncation error, but with the exception of the trick (4) as now the underlying PDE is (8) and not (7):

$$\begin{aligned} u_{tt} = (u_t)_t &= (-au_x + \frac{a\Delta x}{2}(1-a\lambda)u_{xx})_t \\ &= -a(u_t)_x + \mathcal{O}(\Delta x) \\ &= -a(-au_x + \frac{a\Delta x}{2}(1-a\lambda)u_{xx})_x + \mathcal{O}(\Delta x) \\ &= a^2 u_{xx} + \mathcal{O}(\Delta x). \end{aligned} \tag{9}$$

Plugging (9) and the PDE (8) into the computation of L yields

$$\begin{aligned} L &\overset{(3a)}{=} u_t + \frac{\Delta t}{2} u_{tt} + \frac{\Delta t^2}{6} u_{ttt} \\ &\quad + a u_x - \frac{a\Delta x}{2} u_{xx} + \frac{a\Delta x^2}{6} u_{xxx} + \mathcal{O}(\Delta x^3) \\ &= \frac{a\Delta x}{2}(1 - a\lambda) u_{xx} + \frac{\Delta t}{2}(a^2 u_{xx} + \mathcal{O}(\Delta x)) - \frac{a\Delta x}{2} u_{xx} + \mathcal{O}(\Delta x^2) \\ &= \frac{a\Delta x}{2}(1 - a\lambda) u_{xx} + a^2 \frac{\Delta t}{2} u_{xx} - \frac{a\Delta x}{2} u_{xx} + \mathcal{O}(\Delta x^2) - \frac{a\Delta x}{2}(1 - a\lambda) u_{xx} \\ &= \mathcal{O}(\Delta x^2). \end{aligned}$$

Problem 3 (Second-order Hyper):

The differences that can be observed between Lax-Wendroff and the upwind scheme are explained in chapter 19 of the script. The upwind scheme is monotone and thus introduces a blurring effect as seen in problem 1, where the sharp discontinuities of the box function are smoothed.

Lax-Wendroff however is a second order scheme and thus accurate for sufficiently smooth solutions (i.e. a smooth wave), but has trouble with oscillations at discontinuities as they appear in the box function.

A remedy for those shortcomings of both method can be found in hybrid scheme that take the best of both worlds while avoiding the problems.

Problem 4 (What is the true solution?)

For fitting values of epsilon, the problems of the Lax-Wendroff scheme that were discovered in problem 3 are attenuated. The new equation with the added term ϵu_{xx} implements the idea of viscosity solutions that is detailed in 16.3 of the script. For large values of ϵ the influence of the diffusion is bigger and leads to more "smearing out" of the solution, but for $\epsilon \to 0$ a weak solution can be achieved that is closer to the true solution than the pure Lax-Wendroff results.