Numerical Algorithms for Visual Computing II 2010/11
Example Solutions for Assignment 6

Problem 1 (Matrix Stability Infusion)

1. The matrix A of the arising matrix notation $U^{n+1} = AU^n$ takes the following form, if the dirichlet boundary conditions are included by including $U_0^n = a_0$, $U_M^{n+1} = a_M$ in the vector, i.e. $U \in \mathbb{R}^M$ and $A \in \mathbb{R}^{M\times M}$:

$$\begin{aligned} A &:= \begin{pmatrix} 1 & & & & \\ a & 1-2a & a & & \\ & \ddots & \ddots & \ddots & \\ & & a & 1-2a & a \\ & & & & 1 \end{pmatrix} \\ a &:= \frac{D\Delta t}{\Delta x^2} \end{aligned}$$

2. The Eigenvalues λ_k of A can be computed manually or with a computer algebra system. An analytical computation yields:

$$\lambda_k = 1 - 4\frac{D\Delta t}{\Delta x^2}\sin^2\left(\frac{k\pi}{2M}\right)$$

If the magnitude of all Eigenvalues (i.e., the spectral radius) is bounded by one, the scheme is stable. Since the squared sin-term is nonnegative, it only has to be ensured that $\lambda_k > -1$ which yields the condition:

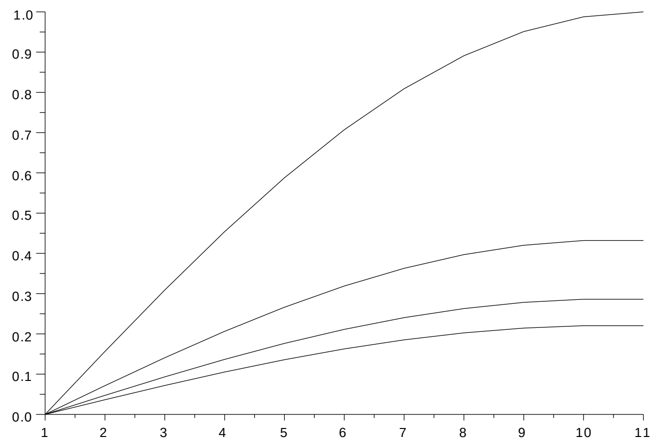
$$\begin{aligned} 0 \le \frac{D\Delta t}{\Delta x^2} &\le \frac{1}{2} \\ \Leftrightarrow \Delta t &\le \frac{\Delta x^2}{2D} \end{aligned}$$

Problem 2 + A7 P 1 (Oscillations: Reality or too much Glogg? + Parabolic Stability Reloaded)

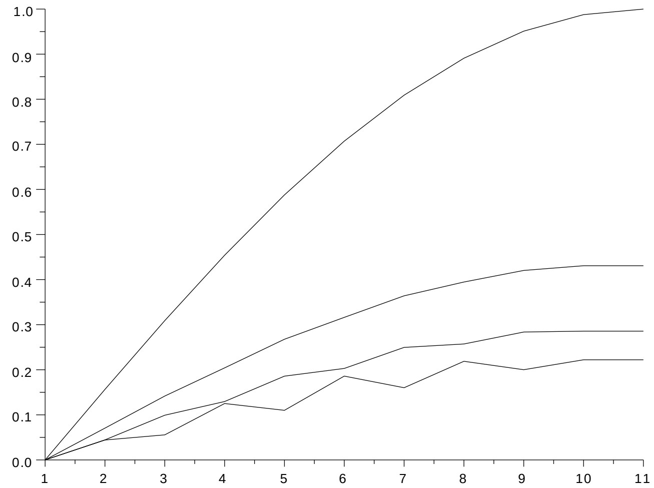
1. Validating the solution consists of checking all four conditions of the IVP:

   - $u_t = e^{-\pi^2 t}\sin\pi x = -\pi^2 e^{-\pi^2 t}\sin\pi x = \frac{\partial}{\partial x}\pi e^{-\pi^2 t}\cos\pi x = \frac{\partial^2}{\partial^2 x}e^{-\pi^2 t}\sin\pi x = u_{xx}$
   - $u(x,0) = e^0\sin\pi x = \sin\pi x$
   - $u(0,t) = e^{-\pi^2 t}\sin 0 = 0$
   - $u(1,t) = e^{-\pi^2 t}\sin\pi = 0$

2. In the first iteration of the exercise, the scheme actually stayed stable for time step values that exceeded the stability condition from problem 1 by far. This stems from the symmetry of the examined sin-function. For all values from the exercise sheet, the results are of the following form:



If one however computes only half of the interval with fitting boundary conditions, as in the second iteration of the exercise (parabolic stability reloaded from assignment 7), oscillations as depicted below occur, if the stability threshold for the time step size is surpassed.



3. The Von Neumann stability analysis yields:

$$\begin{aligned} H(\omega) &= 1 + \frac{D\Delta t}{\Delta x^2}[e^{i\omega\Delta x} - 2 + e^{-i\omega\Delta x}] \\ &= 1 + \frac{D\Delta t}{\Delta x^2}[2cos(\omega\Delta x) - 2] \\ &= 1 - 2\frac{D\Delta t}{\Delta x^2}[1 - cos(\omega\Delta x)] \end{aligned}$$

   In order to avoid the exponentional amplification of error terms in subsequent iterations, the transfer function H must be bounded by 1. This leads to the same stability constraints as the matrix analysis. The form of the transfer function is also responsible for the oscillations that occur when the stability threshold is exceeded.

Problem 3 (Thetas in the Christmas Stockings):

1. A θ-scheme consists of a weighted average of an implicit and an explicit scheme:

$$u_j^{n+1} = u_j^n + \frac{D\Delta t}{\Delta x^2}\left(\theta(u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}) + (1-\theta)(u_{j+1}^n - 2u_j^n + u_{j-1}^n)\right)$$

   The case $\theta = \frac{1}{2}$ is called the Crank-Nicolson method. Its local truncation error is smaller in comparison to the implicit ($\theta = 1$) or explicit

($\theta = 0$) schemes. The space discretisations have a discretisation error of order $\mathcal{O}(\Delta x^2)$ since they represent a standard second order discretisation that was already analysed in the lecture. This is the same in all of the three methods.

However, Crank Nicholson approximates the time derivative at the mesh point $n + \frac{1}{2}$ due to the averaging of the implicit and explicit terms. This in turn increases the order of the truncation error of the forward difference in the time domain from $\mathcal{O}(t)$ to $\mathcal{O}(t^2)$ as proven in assignment 2 Ex. 3.

Hyperbolic Slide into 2011

1. As in problem 3, the local truncation error can be analyzed on the basis of already proven properties of finite difference discretisations. The local truncation error in Δt stays the same in all of the schemes, since all of them use a forward difference, i.e. we have a first order approximation of the time derivatives. Forward or backward differences in the spatial domain lead to a local truncation error of the form $\mathcal{O}(\Delta x)$ while central differences yield a second order approximation in space. Thus, in total, the following truncation errors emerge:

   - backward difference in space: $\mathcal{O}(\Delta t + \Delta x)$
   - forward difference in space: $\mathcal{O}(\Delta t + \Delta x)$
   - central difference in space: $\mathcal{O}(\Delta t + \Delta x^2)$

2. The exercise is concluded with a Von Neumann analysis of the schemes:

   - backward difference:

$$
\begin{aligned}
& |H(\omega)| = |(1 + \frac{a\Delta t}{\Delta x}) * e^0 - \frac{a\Delta t}{\Delta x} e^{-i\omega\Delta x}| \leq 1 \\
\Leftrightarrow \quad & |1 + \frac{a\Delta t}{\Delta x}(1 - e^{-i\omega\Delta x})| \leq 1 \\
\text{worst case:} \quad & 1 + 2\frac{a\Delta t}{\Delta x} \leq 1
\end{aligned}
$$

   Obviously, this cannot be fulfilled, since $a > 0$. Forward differences can be analsysed analogously.

- central difference:

$$
\begin{aligned}
& |H(\omega)| = |1 * e^0 + \frac{a\Delta t}{\Delta x}(e^{i\omega\Delta x} - e^{-i\omega\Delta x})| \leq 1 \\
\Leftrightarrow \quad & |1 - \frac{a\Delta t}{\Delta x}(2\cos\omega\Delta x)| \leq 1 \\
\text{worst case:} \quad & 1 + \frac{2a\Delta t}{\Delta x} \leq 1 \\
\Leftrightarrow \quad & \frac{2a\Delta t}{\Delta x} \leq 0
\end{aligned}
$$