Numerical Algorithms for Visual Computing II 2010/11
Example Solutions for Assignment 5

Problem 1 (How good is trivial?)

1. Remark: This method is also known as the Richardson method and is also employed in some applications, as it may outperform other splitting methods (depending on the used system matrix).

   As we set $N = I$, we can write $A = I - (I - A)$, i.e. the iteration matrix can be computed as

$$\begin{aligned} & Ax &&= b \\ \Leftrightarrow\quad & (I - (I - A))x &&= b \\ \Leftrightarrow\quad & x &&= \underbrace{(I - A)}_{=:M} x + b. \end{aligned}$$

   The iteration scheme therefore is

$$x_{m+1} = \phi(x_m, b) = (I - A)x_m + Ib$$

2. The eigenvalues can be computed by solving $\det(M - \lambda I) = 0$. This gives the solutions $\lambda_1 = 0.1$ and $\lambda_2 = 0.7$. The spectral radius of M, i.e. $\rho(M) = 0.7 < 1$, which shows that the iteration matrix converges according to Theorem 9.0.3.

3.

m	$x_{m,1}$	$x_{m,2}$	ε_m
0	21	-19	20
10	0.8116832	0.8116832	0.188317
20	0.9946805	0.9946805	0.005319
30	0.9998497	0.9998497	0.000150
40	0.9999958	0.9999958	0.000004
50	0.9999999	0.9999999	0.000000
60	1	1	0
70	1	1	0

Problem 2 (How good is Jacobi?)

1. We use already given formula for the iterative Jacobi scheme given by equation 10.13, which uses $M = D^{-1}(D - A)$ and $N = D^{-1}$.

$$M = D^{-1}(D - A) \quad = \quad \begin{pmatrix} 0 & \frac{4}{7} \\ \frac{2}{5} & 0 \end{pmatrix}$$

The eigenvalues of the iteration matrix are $\lambda_{1,2} = \pm\sqrt{\frac{8}{15}}$. The spectral radius can be computed from that by $\rho(M) = \sqrt{\frac{8}{15}} \approx 0.4781$.

2.

m	$x_{m,1}$	$x_{m,2}$	ε_m
0	21	-19	20
10	1.0124778	0.9875222	0.012478
15	0.9996275	1.0002608	0.000373
20	1.0000078	0.9999922	0.000008
25	0.9999998	1.0000002	0.000000
30	1	1	0
35	1	1	0

3. In comparison, the amount of iterations is as double as fast as the standard approach and the spectral radius has been reduced as well. This may lead to the assumption that the spectral radius gives information on the speed of convergence of the method. This can be seen as $\rho(M_{\text{Jacobi}}) \approx (\rho(I - A))^2$.

Problem 3 (How good is Gauß-Seidel?)

1. We use already given formula for the iterative Gauss-Seidel scheme given by equation 10.19, which uses $M = -(D + L)^{-1}R$ and $N = (D + L)^{-1}$.

$$M = -(D + L)^{-1}R \quad = \quad \begin{pmatrix} 0 & \frac{4}{7} \\ 0 & \frac{8}{35} \end{pmatrix}$$

The eigenvalues of M can be computed as $\lambda_1 = 0$ and $\lambda_2 = \frac{8}{35}$. From this we can compute $\rho(M) = \frac{8}{35} \approx 0.22857$.

2.

m	$x_{m,1}$	$x_{m,2}$	ε_m
0	21	-19	20
5	0.9688054	0.9875222	0.031195
10	0.9999805	0.9999922	0.000019
15	1	1	0
20	1	1	0
25	1	1	0

3. One can see that the spectral radius of both iteration methods can be seen as $\rho_{\mathrm{GS}} = (\rho_{\mathrm{J}})^2$. Furthermore one can see that the method also convergence twice as fast.

Problem 4 (How good is SOR?)

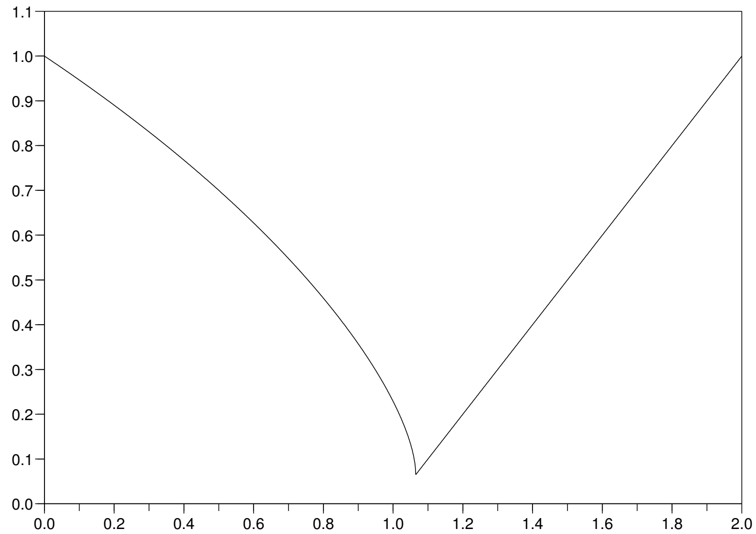
1. The optimal relaxation parameter is for $p = \rho(M_{\mathrm{J}}) = \sqrt{\frac{8}{35}}$. This gives

$$\omega_{\text{optimal}} = \frac{2}{1 + \sqrt{1 - p^2}} \approx 1.0647869$$

2. The eigenvalues of the matrix

$$M = (D + \omega L)^{-1}[(1-\omega)D - \omega R] = \begin{pmatrix} -0.647869 & 0.6084497 \\ -0.0275937 & 0.1943608 \end{pmatrix}$$

for the optimal value for ω are $\lambda_1 = 0.0647107$ and $\lambda_2 = 0.0648632$. This gives us the spectral radius $\rho(M) = 0.0648632$.

3. 

   One can see that for $\omega = 1.0$ which is equal to the standard Gauss-Seidel method, the convergence speed is close to that one of the SOR method, however with $\omega_{optimal}$ one can guarantee a fast convergence to the solution.

4.

m	$x_{m,1}$	$x_{m,2}$	ε_m
0	21	-19	20
5	0.9987226	0.9997003	0.001277
10	1	1	0
15	1	1	0

5. Experimentally, the Gauss-Seidel method error becomes (numerically) 0 after 16 iterations, while the SOR method needs only 9 iterations. Similarly to the exercises before, we can see that $\rho(M_{\mathrm{SOR}}) \approx \rho(M_{\mathrm{GS}})^2$ which resulted in nearly twice as fast results that converge to the result. From that result it is quite obvious that one should achieve finding an iterative matrix with minimal eigenvalue in order to assure fast convergence to the result.

Problem 5 (How good are preconditioners?)

1. According to definition 12.2.1, for a splitting method $x_{j+1} = Mx_j + Nb$ the associated (left) preconditioner P is equal to N. Thus we get the following preconditioners:

splitting method	P	concrete values of P
Jacobi	D^{-1}	$\begin{pmatrix} 1.429 & 0 \\ 0 & 2 \end{pmatrix}$
Gauß-Seidel	$(D+L)^{-1}$	$\begin{pmatrix} 1.429 & 0 \\ 0.571 & 2 \end{pmatrix}$
SOR	$\omega(D+\omega L)^{-1}$	$\begin{pmatrix} 1.521 & 0 \\ 0.648 & 2.13 \end{pmatrix}$

2. Using a left preconditioner P in one of the iterative schemes comes down to solving $PA = Pb$ using the corresponding method. We examine the condition number of PA in comparison to $\text{cond}(A) \approx 3.166$ for all of our iterative methods:

splitting method	$\text{cond}(PA)$
Jacobi	2.876
Gauß-Seidel	1.99
SOR	1.95

We observe that the condition number of PA is significantly smaller than $\text{cond}(A)$ which implicates a faster convergence of the iterative schemes.