Numerical Algorithms for Visual Computing II 2010/11
Example Solutions for Assignment 4

Problem 1 (Matrix and Matrix Reloaded) This exercise deals with the influence of different linear orderings of 16 computational nodes that are given for the Poisson problem. Let us first consider the usual way of ordering the points: we start at the left node of the topmost row of nodes and traverse it to the right, then we move to the leftmost point of the next row, just as we would read text. Thus, $u_1, \ldots, u_4$ are the nodes of the first (counting from the top) row, $u_5, \ldots, u_8$ form the second row and so on. The resulting linear ordering is:

$$(u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6 \quad u_7 \quad u_8 \quad u_9 \quad u_{10} \quad u_{11} \quad u_{12} \quad u_{13} \quad u_{14} \quad u_{15} \quad u_{16})^\top \qquad (1)$$

and with the underlying process

$$-\left[\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{\Delta x^2} + \frac{u_{i,j+1} - 2u_{ij} + u_{i,j-1}}{\Delta y^2}\right] = f_{ij} \qquad (2)$$

we get the following coefficient matrix:

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
4 & -1 & & & -1 & & & & & & & & & & & \\
-1 & 4 & -1 & & & -1 & & & & & & & & & & \\
& -1 & 4 & -1 & & & -1 & & & & & & & & & \\
& & -1 & 4 & & & & -1 & & & & & & & & \\
\hline
-1 & & & & 4 & -1 & & & -1 & & & & & & & \\
& -1 & & & -1 & 4 & -1 & & & -1 & & & & & & \\
& & -1 & & & -1 & 4 & -1 & & & -1 & & & & & \\
& & & -1 & & & -1 & 4 & & & & -1 & & & & \\
\hline
& & & & -1 & & & & 4 & -1 & & & -1 & & & \\
& & & & & -1 & & & -1 & 4 & -1 & & & -1 & & \\
& & & & & & -1 & & & -1 & 4 & -1 & & & -1 & \\
& & & & & & & -1 & & & -1 & 4 & & & & -1 \\
\hline
& & & & & & & & -1 & & & & 4 & -1 & & \\
& & & & & & & & & -1 & & & -1 & 4 & -1 & \\
& & & & & & & & & & -1 & & & -1 & 4 & -1 \\
& & & & & & & & & & & -1 & & & -1 & 4
\end{array}\right]$$

Note that the block form of the matrix arises from the boundary conditions, just as described in the lecture notes.

1. We now represent the ordering defined by table (a) in terms of the naming conventions established in equation 1:

$$(u_1 \quad u_3 \quad u_6 \quad u_{10} \quad u_2 \quad u_5 \quad u_9 \quad u_{13} \quad u_4 \quad u_8 \quad u_{12} \quad u_{15} \quad u_7 \quad u_{11} \quad u_{14} \quad u_{16})^\top \tag{3}$$

This yields the following matrix:

$$\left[\begin{array}{|c|cc|ccc|cccc|ccc|cc|c|}
\hline
4 & -1 & -1 & & & & & & & & & & & & & \\
\hline
-1 & 4 & & -1 & -1 & & & & & & & & & & & \\
-1 & & 4 & & -1 & -1 & & & & & & & & & & \\
\hline
 & -1 & & 4 & & & -1 & -1 & & & & & & & & \\
 & -1 & -1 & & 4 & & & -1 & -1 & & & & & & & \\
 & & -1 & & & 4 & & & -1 & -1 & & & & & & \\
\hline
 & & & -1 & & & 4 & & & & -1 & & & & & \\
 & & & -1 & -1 & & & 4 & & & -1 & -1 & & & & \\
 & & & & -1 & -1 & & & 4 & & & -1 & -1 & & & \\
 & & & & & -1 & & & & 4 & & & -1 & & & \\
\hline
 & & & & & & -1 & -1 & & & 4 & & & -1 & & \\
 & & & & & & & -1 & -1 & & & 4 & & -1 & -1 & \\
 & & & & & & & & -1 & -1 & & & 4 & & -1 & \\
\hline
 & & & & & & & & & & -1 & -1 & & 4 & & -1 \\
 & & & & & & & & & & & -1 & -1 & & 4 & -1 \\
\hline
 & & & & & & & & & & & & & -1 & -1 & 4 \\
\hline
\end{array}\right]$$

2. In the same way, we will first give the alternate description of the reordering defined by table (b):

$$(u_1 \quad u_{12} \quad u_{16} \quad u_{10} \quad u_6 \quad u_5 \quad u_{14} \quad u_2 \quad u_4 \quad u_8 \quad u_9 \quad u_{15} \quad u_{13} \quad u_7 \quad u_3 \quad u_{11})^\top \tag{4}$$

and get the corresponding matrix:

4	0	0	0	0	−1	0	0	0	0	0	−1	0	0	0	0
0	4	0	0	0	0	0	0	0	−1	0	0	0	−1	−1	0
0	0	4	0	0	0	−1	0	−1	0	−1	0	0	0	0	0
0	0	0	4	0	−1	0	−1	0	0	0	0	−1	0	0	0
0	0	0	0	4	−1	0	−1	0	0	0	−1	0	−1	0	0
−1	0	0	−1	−1	4	0	0	0	0	0	0	0	0	0	0
0	0	−1	0	0	0	4	−1	0	0	0	0	−1	0	0	0
0	0	0	−1	−1	0	−1	4	−1	0	0	0	0	0	0	0
0	0	−1	0	0	0	0	−1	4	0	0	0	0	−1	−1	0
0	−1	0	0	0	0	0	0	0	4	0	0	0	0	0	−1
0	0	−1	0	0	0	0	0	0	0	4	0	0	0	−1	0
−1	0	0	0	−1	0	0	0	0	0	0	4	0	0	0	−1
0	0	0	−1	0	0	−1	0	0	0	0	0	4	0	0	0
0	−1	0	0	−1	0	0	0	−1	0	0	0	0	4	0	−1
0	−1	0	0	0	0	0	0	−1	0	−1	0	0	0	4	0
0	0	0	0	0	0	0	0	0	−1	0	−1	0	−1	0	4

Problem 2 (Crossing derivatives)

1. We have been given the following cross derivative discretisation by use of central difference methods

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial}{\partial y} &= \frac{\partial}{\partial x}\left(\frac{u_{i,j+1} - u_{i,j-1}}{2h}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{u_{i,j+1}}{2h}\right) - \frac{\partial}{\partial x}\left(\frac{u_{i,j-1}}{2h}\right) \\
&= \frac{u_{i+1,j+1} - u_{i-1,j+1}}{4h^2} - \frac{u_{i+1,j-1} - u_{i-1,j-1}}{4h^2} \\
&= \frac{1}{4h^2}(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1})
\end{aligned}$$

with $h = \Delta x = \Delta y$. For this 4 points we can compute the 2D-Taylorexpansion

$$\begin{aligned}
u(x_0, y_0) &= f(x,y) + \binom{1}{0}\frac{\partial}{\partial x} f(x,y)(x-x_0) + \binom{1}{1}\frac{\partial}{\partial y} f(x,y)(y-y_0) \\
&+\frac{1}{2}\left(\binom{2}{0}\frac{\partial^2}{\partial x^2} f(x,y)(x-x_0)^2 + \binom{2}{1}\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y)(x-x_0)(y-y_0)\right. \\
&+\left.\binom{2}{2} f(x,y)\frac{\partial^2}{\partial y^2}(y-y_0)^2\right)
\end{aligned}$$

This gives for our simple points the following approximation:

$$\begin{aligned}
u(x \pm h, y+h) = u &\pm hu_x + u_y + \frac{1}{2}h^2(u_{xx} \pm 2u_{xy} + u_{yy}) \\
&+ \frac{1}{6}h^3(\pm u_{xxx} + 3u_{xxy} \pm u_{xyy} + 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} \pm 4u_{xxxy} + 6u_{xxyy} \pm 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5) \\
u(x+h, y-h) = u &+ hu_x - u_y + \frac{1}{2}h^2(u_{xx} - 2u_{xy} + u_{yy}) \\
&+ \frac{1}{6}h^3(u_{xxx} - 3u_{xxy} + u_{xyy} - 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5) \\
u(x-h, y-h) = u &- hu_x - u_y + \frac{1}{2}h^2(u_{xx} + 2u_{xy} + u_{yy}) \\
&- \frac{1}{6}h^3(u_{xxx} + 3u_{xxy} + u_{xyy} + 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5)
\end{aligned}$$

Now we can input this approximation into our four pixel scheme:

$$\begin{aligned}
u_{xy} &\approx \frac{1}{4h^2}(u_{i+1,j+1} - u_{i-1,j+1} - u_{i+1,j-1} + u_{i-1,j-1}) \\
&= \frac{1}{4h^2}(u + hu_x + u_y + \frac{1}{2}h^2(u_{xx} + 2u_{xy} + u_{yy}) \\
&+ \frac{1}{6}h^3(u_{xxx} + 3u_{xxy} + u_{xyy} + 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5) \\
&\quad -(u - hu_x + u_y + \frac{1}{2}h^2(u_{xx} - 2u_{xy} + u_{yy}) \\
&+ \frac{1}{6}h^3(-u_{xxx} + 3u_{xxy} - u_{xyy} + 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5)) \\
&\quad -(u + hu_x - u_y + \frac{1}{2}h^2(u_{xx} - 2u_{xy} + u_{yy}) \\
&+ \frac{1}{6}h^3(u_{xxx} - 3u_{xxy} + u_{xyy} - 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5)) \\
&\quad +u - hu_x - u_y + \frac{1}{2}h^2(u_{xx} + 2u_{xy} + u_{yy}) \\
&- \frac{1}{6}h^3(u_{xxx} + 3u_{xxy} + u_{xyy} + 3u_{yyy}) \\
&+ \frac{1}{24}h^4(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}) + \mathcal{O}(h^5))
\end{aligned}$$

If we combine this terms together, we will get:

$$\begin{aligned}
\frac{1}{4h^2} \quad & (u \underbrace{(1-1-1+1)}_{=0} + hu_x \underbrace{(1+1-1-1)}_{=0} + hu_y \underbrace{(1-1+1-1)}_{=0} \\
+ \quad & \frac{1}{2}h^2 \underbrace{(u_{xx} + 2u_{xy} + u_{yy} - u_{xx} + 2u_{xy} - u_{yy} - u_{xx} + 2u_{xy} - u_{yy} + u_{xx} + 2u_{xy} + u_{yy})}_{8u_{xy}} \\
+ \quad & \frac{1}{6}h^3 (u_{xxx} \underbrace{(1+1-1-1)}_{=0} + 3u_{xxy} \underbrace{(1-1+1-1)}_{=0} \\
+ \quad & 3u_{xyy} \underbrace{(1+1-1-1)}_{=0} + u_{yyy} \underbrace{(1-1+1-1)}_{=0}) \\
+ \quad & \frac{1}{24}h^4 (u_{xxxx} \underbrace{(1-1-1+1)}_{=0} + 4u_{xxxy} \underbrace{(1+1+1+1)}_{=4} \\
& +6u_{xxyy} \underbrace{(1-1-1+1)}_{=0} + 4u_{xyyy} \underbrace{(1+1+1+1)}_{=4} + u_{yyyy} \underbrace{(1-1-1+1)}_{=0}) + \mathcal{O}(h^5))
\end{aligned}$$

This sums up to

$$\begin{aligned}
& \frac{1}{4h^2}(4h^2 u_{xy} + \frac{16}{24}(u_{xxxy} + u_{xyyy}) + \mathcal{O}(h^5)) \\
= \quad & u_{xy} + \frac{1}{6}h^2(u_{xxxy} + u_{xyyy}) + \mathcal{O}(h^3) \\
= \quad & u_{xy} + \frac{1}{6}h^2 \Delta u_{xy} + \mathcal{O}(h^3) \\
= \quad & \left(1 + \frac{1}{6}h^2\right) u_{xy} + \mathcal{O}(h^3) \\
\Rightarrow \quad & \mathcal{O}(h^2)
\end{aligned}$$

Overall we get a $\mathcal{O}(h^2)$ error term for the cross derivative approximation.

2. This discretisation is isotropic, as the error term incorporates an additional isotropic Laplace operator onto u_{xy}, which we wanted to approximante in the first place.

Problem 3 (Sobel Operator: Condition and Precision)

1. If we write the Gaussian convolution with the central difference as a single equation we get:

   $u_x \approx \frac{1}{8h}(u_{i+1,j+1} + 2u_{i+1,j} + u_{i+1,j-1} - u_{i-1,j-1} - 2u_{i-1,j} - u_{i-1,j-1})$

   Plugging in the tailor expansion of all of the discrete points that form the equation yields:

   $$\begin{aligned} \frac{1}{8h} \quad (& \quad (-1+1-2+2-1+1)u \\ + & \quad h(1+1+2+2+1+1)u_x + h(1-1-1+1)u_y \\ + & \quad \frac{h^2}{2}(-1+1-2+2-1+1)u_{xx} + h^2(1-1-1+1)u_{xy} \\ + & \quad \frac{h^2}{2}(-1+1-1+1)u_{yy} \\ + & \quad \frac{h^3}{6}(1+1+2+2+1+1)u_{xxx} + \frac{h^3}{2}(1-1-1+1)u_{xxy} \\ + & \quad \frac{h^3}{6}(1+1+1+1)u_{xyy} + \frac{h^3}{2}(1-1-1+1)u_{yyy} \\ + & \quad \mathcal{O}(h^4)) \end{aligned}$$

   Thus, we finally get:

   $$u_x \approx u_x + \frac{h^2}{12}(2u_{xxx} + 3u_{xyy}) + \mathcal{O}(h^3)$$

   which means the local truncation error has order 2.

2. Since the leading error term is - up to a scalar factor in the mixed derivative - identical to $\partial_x \Delta u$, we consider the discretisation isotropic.

Problem 4 (Is this stencil good™ or evil™?)

1. The equation corresponding to the stencil is

   $$\begin{aligned} \Delta u \quad \approx & \quad \alpha(u_{i+1,j+1} + u_{i+1,j-1} - 4u_{i,j} + u_{i-1,j-1} + u_{i-1,j-1} \\ + & \quad u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) \end{aligned}$$

   As in the previous exercise, we plug in the Taylor-expansions and get

   $$\Delta u \approx \alpha(\frac{h^2 \Delta u}{6} + \frac{h^4}{24}(u_{xxxx} + 6u_{xxyy} + u_{yyyy}) + \mathcal{O}(h^3)$$

   which means the local truncation error has order 2.

2. There are several noteworthy observations to make: Obviously, the approximation is not consistent and a scalar factor of $\frac{1}{2h^2}$ instead of $\frac{1}{4h^2}$ would make the approximation consistent.

   Additionally, there is a bias in direction of the mixed derivative, but since it is scalar-valued, we still consider the discretisation to be isotropic.

   Another important aspect however is the reducibility of the matrix. We give an alternative criterion that helps to identify irreducible matrices by only looking at the corresponding stencil: if there is a "path" of positive stencil weights, that connects all pairs (i, j) of nodes, the matrix is irreducible. In other words, you start at any node i with the stencil centered on this node. You may now move to any adjacent node, that has a positive value in the stencil. In the case of the stencil in this exercise, you might move in all four diagonal directions, since the stencil weight is 1 there and 0 everywhere else. You are now at a new node k. Consider again a stencil centered at k - you now have again a choice of directions to take. By moving along stencil weights and moving the stencil along, you have to find a path to any arbitrary other node j.

   In the context of this particular stencil, as already mentioned, only diagonal movements are possible. Thus, it's like diagonal movement on a chess board: if you start on a black (white) field, you can only reach other black (white) fields. Consequently, the matrix in exercise 4 is reducible.

Problem 5 (Is this stencil good™ or evil™?)

1. The equation corresponding to the stencil is

$$\Delta u \approx \frac{1}{4h^2}(u_{i+1,j+1} + u_{i+1,j-1} - 4u_{i,j} + u_{i-1,j-1} + u_{i-1,j-1})$$

   As in the previous exercise, we plug in the Taylor-expansions and get

$$\Delta u \approx 6h^2 \Delta u + \frac{h^4}{2}(u_{xxxx} + 2u_{xxyy} + u_{yyyy}) + \mathcal{O}(h^3)$$

   If we choose $\alpha = \frac{1}{6h^2}$ which means the local truncation error has order 2.

2. In this case, we have a consistent discretisation that is also isotropic. Additionally, we can move in each direction since all stencil weights except for the center are positive. Thus there is a path from each node i to

every other node j and the matrix is irreducible (see Problem 4). Thus, the stencil from problem 5 is the better choice for the discretisation of the Poisson equation.