Numerical Algorithms for Visual Computing II

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Assignment 8 (2+1 Exercises) – The Final Countdown plus One Sugar

Exercise No. 1 – Characteristic Climax

Consider the IVP for the PDE

$$u_t + f(u)_x = 0 \tag{1}$$

and the so-called *characteristics*: These are the curves $x(t)$ in the x-t-domain along which

$$\frac{d}{dt} u(x(t), t) = 0 \tag{2}$$

holds.

1. Let $f(u) := 2u$. Compute the constituting equation of the characteristics for this case, and sketch the characteristics in the (x, t)-domain. **(3pts)**
2. Let $f(u) := \frac{1}{2}u$. Compute the constituting equation of the characteristics for this case, and sketch the characteristics in the (x, t)-domain. **(3pts)**
3. Let $f(u) := \frac{1}{2}u^2$ and let the following initial condition be given:

$$u_0(x) := \begin{cases} 1 & : \quad x \leq 0 \\ 1 - x & : \quad 0 \leq x \leq 1 \\ 0 & : \quad x \geq 1 \end{cases} \tag{3}$$

   Compute the constituting equation of the characteristics for this case, and sketch the characteristics in the (x, t)-domain up to $t = 1$. Discuss what happens at $(x, t) = (1, 1)$. **(6pts)**
4. Let again be $f(u) := au$, for $a := 2$. Consider the characteristics in the x-t-domain, and assume in addition the presence of a spatial grid with mesh width $h := 0.5$.

   Consider then also the upwind scheme

$$U_j^{n+1} = U_j^n - a \frac{\Delta t}{\Delta x} \left[U_j^n - U_{j-1}^n \right]$$

   and the corresponding CFL-condition. Sketch the characteristics together with the spatial grid in the x-t-domain, and determine graphically a stable time step size. Discuss the relation of what you found to the CFL-condition. **(6pts)**

Exercise No. 2 – Parabolic Recall

Consider the Perona-Malik-model for nonlinear isotropic diffusion filtering

$$u_t = \nabla \cdot (D \nabla u) \tag{4}$$

where D is a diffusion tensor: It is of the form $D = g(|\nabla u|^2)I$, where I is the 2×2 identity matrix, and g is a given by

$$g(s^2) \quad = \quad \frac{1}{1 + \frac{s^2}{\lambda^2}} \ , \quad \lambda > 0 \tag{5}$$

For any computation, set $\lambda := 2$ in the following.

1. Construct a numerical solver for the Perona-Malik-model. Discuss in detail the steps you take. **(6pts)**

2. Define discrete boundary conditions that ensure that the average grey value of a given image which initialises the discretisation of (4) is conserved. **(6pts)**

Exercise No. 3 – Discrete Theorem of Gauß

Let us consider the Lax-Wendroff scheme for discretising (1)

$$\begin{aligned} U_j^{n+1} \quad = \quad & U_j^n - \frac{\Delta t}{2\Delta x}\left[f\left(U_{j+1}^n\right) - f\left(U_{j-1}^n\right)\right] \\ & + \frac{\Delta t^2}{2\Delta x^2}\left[A_{j+1/2}\left(f\left(U_{j+1}^n\right) - f\left(U_j^n\right)\right) - A_{j-1/2}\left(f\left(U_j^n\right) - f\left(U_{j-1}^n\right)\right)\right] \end{aligned} \tag{6}$$

where

$$\begin{aligned} A_{j+1/2} \quad &:= \quad f'\left(\frac{1}{2}\left(U_{j+1}^n + U_j^n\right)\right) \\ A_{j-1/2} \quad &:= \quad f'\left(\frac{1}{2}\left(U_j^n + U_{j-1}^n\right)\right) \end{aligned} \tag{7}$$

Write the scheme in conservation form, determining the numerical flux function g.

(10 EXTRA points!)