Numerical Algorithms for Visual Computing II

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Assignment 7 (4 Exercises) – Hyperbolic New Year

Exercise No. 1 – $\overset{Paranormal..?}{\textbf{Parabolic}}$ Stability Reloaded

As for Assignment 6, consider the scheme

$$U_j^{n+1} \; = \; U_j^n + \frac{D\Delta t}{\Delta x^2} \left[U_{j+1}^n - 2U_j^n + U_{j-1}^n \right] \tag{1}$$

for linear diffusion

$$u_t \; = \; D \cdot u_{xx} \; , \quad D > 0 \tag{2}$$

For Assignment 6, we employed as *Test Problem No. 1* the parameter choice $D := 1$, and we defined the following set of initial and boundary conditions:

$$\begin{aligned} u(x,0) &= \sin \pi x \\ u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}$$

Now, let us simplify the above set-up. One can realise that the problem has a symmetry over the interval $[0,1]$: The solution over $[0,0.5]$ may just be mirrored at $x = 0.5$ to obtain the solution for $[0.5,1]$. Therefore, in order to save computational effort (here not an issue, but for larger problems it is!) we just aim for the numerical solution over $[0,0.5]$. Therefore, we employ the new set of initial and boundary conditions defined as:

$$\begin{aligned} u(x,0) &= \sin \pi x \qquad \text{over } [0,0.5] \\ u(0,t) &= 0 \\ \frac{\partial}{\partial x} u(0.5,t) &= 0 \end{aligned}$$

The last condition encodes that a maximum of u is at $x = 0.5$, and it is also compatible to the symmetry. We cannot use Dirichlet conditions as we do not know $u(0.5,t)$ from the original problem.

Code the method (1) for *Test Problem No. 1*, using $\Delta x = 0.05$. To this end, the *Neumann boundary condition* at $x = 0.5$ should be implemented by means of a 'ghost point' realising $u(0.5,t) = u(0.5 + \Delta x, t)$.

As for Assignment 6, perform two sets of calculations. For the first one use $\Delta t^{(1)} := 0.001125$. Plot the results at

$$\begin{aligned} t &= 0.07425 \\ t &= 0.111375 \\ t &= 0.1485 \end{aligned}$$

together with the exact solution.
For the second one use $\Delta t^{(2)} := 0.001375$. Plot the results at

$$\begin{aligned} t &= 0.07425 \\ t &= 0.111375 \\ t &= 0.136125 \end{aligned}$$

together with the exact solution. Also plot the initial condition in both cases. Perform the plots over the whole, original domain $[0, 1]$.

1. Compare and discuss the results. **(3 pts)**
2. Explain the shape of posssible oscillations making use of the results of the von Neumann stability analysis. **(3 pts)**

Exercise No. 2 – Hyper, Hyper!

Consider the tranport ('linear advection') equation

$$u_t = au_x = 0\ , \quad a > 0 \tag{3}$$

and the *upwind scheme*

$$U_j^{n+1} = U_j^n - a\frac{\Delta t}{\Delta x}\left[U_j^n - U_{j-1}^n\right]$$

1. Write a program solving (3) via the upwind scheme. Use the parameters $a = 2, \Delta x = 0.1, \Delta t = 0.025$, and perform 20 time steps iterating

   $$U_i^0 = \begin{cases} 1, & i = 0, \ldots, 9, \\ 0, & \text{else}, \end{cases} \tag{4}$$

   where $U_i^0 \approx u(i\Delta x, 0)$. Plot your result.

2. Compute the local truncation error of the upwind scheme.

   Hint: You need Taylor series expansions in the time variable t as well as in the space variable x. These work both as usual.

3. Verify the relationship

$$u_{tt} = a^2 u_{xx} \tag{5}$$

by use of the PDE (3).

4. Compute the leading error term in the format

$$L(u) = \alpha(\Delta x) \cdot u_{xx} + \mathcal{O}((\Delta x)^2), \tag{6}$$

i.e. derive the function $\alpha(\Delta x)$.

Hints:

- $\lambda := \frac{\Delta t}{\Delta x} = \text{constant}$
- $\mathcal{O}((\Delta t)^2) = \mathcal{O}((\Delta x)^2)$, since $\Delta t = \lambda \cdot \Delta x$
- Use equation (5)

Can you explain the shape of your numerical solution obtained in part (a) by the form of $L(u)$?

5. The so-called modified equation of the upwind scheme for approximating (3) reads as the PDE

$$u_t + a u_x = \frac{a\Delta x}{2}(1 - a\lambda) u_{xx}. \tag{7}$$

Verify, that the upwind scheme gives a second-order accurate approximation of this PDE.

Hint: You need to make use of the latter PDE instead of (3) within the computation.

(8 pts)

Exercise No. 3 – Second-order Hyper

Let us define now in general the *Lax-Wendroff scheme*

$$\begin{aligned} U_j^{n+1} &= U_j^n - \frac{\Delta t}{2\Delta x}\left[f\left(U_{j+1}^n\right) - f\left(U_{j-1}^n\right)\right] \\ &\quad + \frac{\Delta t^2}{2\Delta x^2}\left[A_{j+1/2}\left(f\left(U_{j+1}^n\right) - f\left(U_j^n\right)\right) - A_{j-1/2}\left(f\left(U_j^n\right) - f\left(U_{j-1}^n\right)\right)\right] \end{aligned} \tag{8}$$

where

$$\begin{aligned} A_{j+1/2} &:= f'\left(\frac{1}{2}\left(U_{j+1}^n + U_j^n\right)\right) \\ A_{j-1/2} &:= f'\left(\frac{1}{2}\left(U_j^n + U_{j-1}^n\right)\right) \end{aligned} \tag{9}$$

Solve again numerically the linear advection equation (3) defined via $f(u) = au$ (linear flux!) with $a > 0$. For this, make use of the upwind scheme and the Lax-Wendroff scheme.

For this, use two types of initial conditions:

(i) Over $[-1, 5]$, make use of $u(x, 0) = 1$ for $x \in [0, 1]$, $u(x, 0) = 0$ else.

(ii) Employ $u(x, 0) = \sin \pi x$ over $[0, \pi]$.

While the interval in the first setting should be large enough to contain all evolutions without needing specific boundary conditions, you should employ periodic boundary conditions over $[0, \pi]$, i.e. $u(0, t) = u(\pi, t)$. In the latter setting it is a good idea to let the evolution of the sine wave run up to stopping times where the numerical solution should match the initial condition exactly.

Play around with numerical parameters and report on useful, stable results. What are the main features of the upwind scheme and the Lax-Wendroff scheme? How do the schemes relate to each other? **(10 pts)**

Exercise No. 3 – What is the true solution?

Now, employ the Lax-Wendroff scheme for solving the Buckley-Leverett equation

$$u_t + f(u)_x = 0, \qquad f(u) := \frac{u^2}{u^2 + 0.5(1-u)^2} \tag{10}$$

As initial condition, use again the definition from (4).

After this initial computation, add on the right hand side of the PDE (10) a small diffusion ϵu_{xx}. Discretise this term in addition. Play around with various values of epsilon, and compare your results with the ones from the pure Lax-Wendroff discretisation withour additional diffusion term.

Did you expect your results? Give a discussion. **(6 pts)**