Numerical Algorithms for Visual Computing II

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Assignment 6 (4 Exercises) – Christmas Biscuits

Consider the scheme

$$U_j^{n+1} \;=\; U_j^n + \frac{D\Delta t}{\Delta x^2}\left[U_{j+1}^n - 2U_j^n + U_{j-1}^n\right] \tag{1}$$

for linear diffusion

$$u_t \;=\; D \cdot u_{xx}\;, \quad D > 0 \tag{2}$$

As *Test Problem No. 1* used for some of the exercises we set $D := 1$, and we define the following set of initial and boundary conditions:

$$\begin{aligned} u(x,0) &= \sin \pi x \\ u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}$$

Exercise No. 1 – Matrix Stability Infusion

Consider the scheme

$$U_j^{n+1} \;=\; U_j^n + \frac{D\Delta t}{\Delta x^2}\left[U_{j+1}^n - 2U_j^n + U_{j-1}^n\right] \tag{3}$$

for linear diffusion

$$u_t \;=\; D \cdot u_{xx}\;, \quad D > 0 \tag{4}$$

1. For $M+1$ discretisation points and corresponding vectors U^n and U^{n+1}, write the method (3) in matrix format $U^{n+1} = AU^n$. To this end, you may assume Dirichlet boundary conditions $U_0^n := a_0$, $U_M^n := a_M$. **(2 pts)**
2. Compute the eigenvalues of the matrix A. You may try this analytically, or you may employ SciLab. Discuss your findings with respect to stability of the method and with respect to the convergence of the iteration. **(4 pts)**

Exercise No. 2 – Oscillations: Reality or too much Glogg?

1. Validate that the exact solution of *Test Problem No. 1* reads as

$$u(x,t) = e^{-\pi^2 t} \sin \pi x \tag{5}$$

(2 pts)

2. Code the method (3) for *Test Problem No. 1*, using $\Delta x = 0.05$.

   Perform two sets of calculations. For the first one use $\Delta t^{(1)} := 0.001125$. Plot the results at

$$\begin{aligned} t &= 0.07425 \\ t &= 0.111375 \\ t &= 0.1485 \end{aligned}$$

   together with the exact solution.
   For the second one use $\Delta t^{(2)} := 0.001375$. Plot the results at

$$\begin{aligned} t &= 0.07425 \\ t &= 0.111375 \\ t &= 0.136125 \end{aligned}$$

   together with the exact solution. Also plot the initial condition in both cases.

   Compare and discuss the results. **(2 pts)**

3. Explain the shape of posssible oscillations making use of the results of the von Neumann stability analysis. **(2 pts)**

Exercise No. 3 – Thetas in the Christmas Stockings

Devise a θ-scheme for marching in time instead of the pure explicit scheme above.

1. Compute the local truncation error of the resulting θ-scheme for $\theta = 1/2$. **(2 pts)**

2. Code the θ-scheme and validate experimentally the stability criterion

$$\begin{aligned} 2D\frac{D\Delta t}{\Delta x^2} \leq \frac{1}{1-2\theta} \quad &\text{if} \quad 0 \leq \theta < 1/2 \\ \text{no restriction} \quad &\text{if} \quad 1/2 \leq \theta \leq 1 \end{aligned}$$

   making use of the setting from *Test Problem No. 1*. **(4 pts)**

3. Perturb the initial condition from *Test Problem No. 1* by small sine waves

   $C \sin(m\Delta x)$, $C > 0$

   There should be small oscillations visible at the discretisation points, overlaying the initial condition from the *Test Problem No. 1*.

   Start with a small oscillation of the scale of twice the mesh spacing Δx. Apply the methods with $\theta = 0, 1/2, 1$ at the perturbed signal for some time steps and discuss the results with respect to the damping of the oscialltions and the accuracy of the methods. Play around with number of time steps, wave lengths of perturbations, and comment on what you observe. **(4 pts)**

Exercise No. 4 – Hyperbolic Slide into 2011

Consider the tranport equation

$$u_t = au_x = 0\ , \quad a > 0 \tag{6}$$

and the numerical schemes

$$\begin{aligned}
U_j^{n+1} &= U_j^n - a\frac{\Delta t}{\Delta x}\left[U_j^n - U_{j-1}^n\right] \quad \text{(backward difference in space)}\\
U_j^{n+1} &= U_j^n - a\frac{\Delta t}{\Delta x}\left[U_{j+1}^n - U_j^n\right] \quad \text{(forward difference in space)}\\
U_j^{n+1} &= U_j^n - a\frac{\Delta t}{\Delta x}\left[U_{j+1}^n - U_{j-1}^n\right] \quad \text{(central difference in space)}
\end{aligned}$$

1. Compute the local truncation errors of the methods. Comment your findings. **(2 pts)**

2. Perform the von Neumann stability analysis for the three methods. **(6 pts)**