Numerical Algorithms for Visual Computing II

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Assignment 5 (5 Exercises) – The SOR-Before-Christmas-Chill

We consider for Christmas the very simple model problem

$$\underbrace{\begin{pmatrix} 0.7 & -0.4 \\ -0.2 & 0.5 \end{pmatrix}}_{=:\,\mathbf{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{=:\,\mathbf{x}} = \underbrace{\begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix}}_{=:\,\mathbf{b}} \tag{1}$$

for use for the linear-system-solver stuff.

Note that the exact solution is $(1, 1)^T$. To compute the solution is not the big issue here: We would like to get some insight on how the solvers behave.

For starting corresponding iterations, use $\mathbf{x}_0 = (21, -19)^T$.

For evaluation, use the ∞-Norm: $\|(a, b)^T\|_\infty = \max(|a|, |b|)$ as indicated below.

Exercise No. 1 – How good is trivial?

(a) Set $N = I$, where $I = diag(1, 1)$, and define a consistent linear iterative scheme for solving (1).

(b) Compute at hand of the two eigenvalues the spectral radius of the arising iteration matrix.

(c) Code the linear iterative scheme. Evaluate the solution at the iterates $m = 0, 10, 20, 30, 40, 50, 60, 70$. Compute at these stages the error $\epsilon_m = \|\mathbf{x}_m - \mathbf{A}^{-1}\mathbf{b}\|_\infty$. **(6 pts)**

Exercise No. 2 – How good is Jacobi?

Consider again the model problem (1). Use now the Jacobi-scheme.

(a) Compute the eigenvalues and the spectral radius of the iteration matrix.

(b) Code the Jacobi-scheme. Evaluate the solution at the iterates $m = 0, 10, 15, 20, 25, 30, 35$. Compute at these stages the error $\epsilon_m = \|\mathbf{x}_m - \mathbf{A}^{-1}\mathbf{b}\|_\infty$.

(c) Compare the spectral radius of the Jacobi-scheme with the one of the trivial scheme from Exercise 1. Can you infer an approximate rule for the gain in convergence efficiency? **(6 pts)**

Exercise No. 3 – How good is Gauß-Seidel?

Consider again the model problem (1). Use now the Gauß-Seidel-scheme.

(a) Compute the eigenvalues and the spectral radius of the iteration matrix.

(b) Code the Gauß-Seidel-scheme. Evaluate the solution at the iterates $m = 0, 5, 10, 15, 20, 25$. Compute at these stages the error $\epsilon_m = \|\mathbf{x}_m - \mathbf{A}^{-1}\mathbf{b}\|_\infty$.

(c) Compare the spectral radius of the Gauß-Seidel-scheme with the one of the Jacobi-scheme from Exercise 1. Can you again infer an approximate rule for the gain in convergence efficiency? **(6 pts)**

Exercise No. 4 – How good is SOR?

Consider again the model problem (1). Use now the SOR-scheme.

(a) Compute the optimal relaxation parameter.

(b) Compute the eigenvalues and the spectral radius of the iteration matrix.

(c) Using the scales $(0, 2)$ for the relaxation parameter and $[0, 1]$ for the spectral radius, plot the grap showing the dependence of the spectral radius from the relaxation parameter. Comparing with other schemes up to now, can you learn something from this on SOR?

(d) Code the SOR-scheme. Evaluate the solution at the iterates $m = 0, 5, 10, 15$. Compute at these stages the error $\epsilon_m = \|\mathbf{x}_m - \mathbf{A}^{-1}\mathbf{b}\|_\infty$.

(e) Compare the spectral radius of the optimal SOR-scheme with the one of the Gauß-Seidel-scheme from Exercise 1. Can you again infer an approximate rule for the gain in convergence efficiency? **(12 pts)**

Exercise No. 5 – How good are preconditioners?

Consider again the model problem (1), and the iterative solvers you have coded up to now.

(a) Determine splitting-based left preconditioners corresponding to the Jacobi, Gauß-Seidel and SOR methods for the model problem.

(b) Compare for all iterative schemes the condition numbers of the iteration matrices, using preconditioners and without on the original matrix. (There is a SciLab command for computing the condition number). Can you observe something?

(6 extra pts for Christmas)