Numerical Algorithms for Visual Computing II

Michael Breuß and Pascal Peter

Released: 29.10.2010
Assigned to: Tutorial at 05.11.2010

Assignment 2 (3 Exercises) – Different Differentials

Exercise No. 1 – Normal or Abnormal (8+2+6=16 points)

Let the surface S of an object in 3-D space be parameterised by

$$\begin{aligned} S \ &: \ \Omega \ \rightarrow \ \mathbb{R}^3 \\ (x,y) \ &\mapsto \ \begin{bmatrix} S_1 \\ S_2 \\ S_3 \end{bmatrix} := \frac{f u(x,y)}{\sqrt{x^2+y^2+f^2}} \left(x, y, -f\right)^\top \end{aligned}$$

where $f > 0$ is a parameter, and where $u(x,y)$ describes the depth (i.e., the z-coordinate) of the object. The task is to derive a formula for the *surface normal*.

Proceed as follows:

1. Compute the Jacobian matrix

$$J\left(S(x,y)\right) \ := \ \begin{bmatrix} \frac{\partial S_1}{\partial x} & \frac{\partial S_1}{\partial y} \\ \frac{\partial S_2}{\partial x} & \frac{\partial S_2}{\partial y} \\ \frac{\partial S_3}{\partial x} & \frac{\partial S_3}{\partial y} \end{bmatrix}$$

2. Validate that the column vectors of J describe a plane in 3-D. Give an argumentation, that they are also tangential to S.

3. Use the cross-product of these two vectors to *verify or disprove* that the surface normal $\mathbf{n} \in \mathbb{R}^3$ to S is

$$\mathbf{n}(x,y) \ = \ \left(f\nabla u - \frac{fu}{x^2+y^2+f^2} \begin{pmatrix} x \\ y \end{pmatrix}, \nabla u \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \frac{f^2 u}{x^2+y^2+f^2} \right)^\top$$

   The symbol $\cdot$ thereby denotes the Euclidean scalar product.

Exercise No. 2 – Potentially Elliptic™ (2+6=8 points)

The so-called potential function u of the 3-D sphere $x^2+y^2+z^2 = a^2$, $a \in \mathbb{R}$, is for $x^2+y^2+z^2 \geq a^2$ defined as

$$u\left(x,y,z\right) \ = \ \frac{4\pi a^3}{3\sqrt{x^2+y^2+z^2}}$$

(a) Give an argumentation, that u and its partial derivatives of first and second order are continuous for $x^2+y^2+z^2 \geq a^2$.

(b) Show that u fulfils the 3-D Laplace equation $\Delta u = \frac{\partial^2 u}{\partial^2 x} + \frac{\partial^2 u}{\partial^2 y} + \frac{\partial^2 u}{\partial^2 z} = 0$

Exercise No. 3 – Law and Orders (6 points)

Recall the backward difference law

$$A \ := \ \frac{U_i - U_{i-1}}{\Delta x}$$

which is a first-order approximation of $u'(i\Delta x)$.

Prove, that the order is also: $A = u'((i-1/2)\Delta x) + \mathcal{O}(\Delta x^2)$.