Numerical Algorithms for Visual Computing II

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Released: 22.10.2010
Assigned to: Tutorial at 29.10.2010

Assignment 1 (4 Exercises) – Basic Basics™ of PDEs

Exercise No. 1 – Typesetting of PDEs (4×2=8 points)

Consider a second-order PDE in two variables as an equation of the form

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu \quad = \quad G,$$

where A, B, C, D, E, F, G can be constants or given functions of x and y. It can be shown that parabolic equations satisfy the property $B^2 - 4AC = 0$, hyperbolic equations $B^2 - 4AC > 0$ and elliptic equations $B^2 - 4AC < 0$.

Categorise the following differential equations with respect to the order, linearity or non-linearity with variable coefficients or not, and the type of the PDE:

(a) $u_t = u_{xx}$

(b) $u_{tt} = u_{xx}$

(c) $u_{xx} + u_{yy} = 0$

(d) $xu_x + yu_y + u^2 = 0$

Exercise No. 2 – Tayloring schemes (4+4=8 points)

Compute

- the Taylor expansion,
- the local truncation error

for an approximation of the second derivative approximation of u, i.e. $u''(x)$, by making use only of the mesh points $(j+2)\Delta x$, $j\Delta x$ and $(j-2)\Delta x$.

Exercise No. 3 – How big is this $\mathcal{O}$? (4×2=8 points)

Let $h \in \mathbb{R}$ and $p, q \in \mathbb{N}$. Prove validity / non-validity of the following assertions, also assuming $p < q$:

$$\begin{aligned}
\mathcal{O}(h^p) + \mathcal{O}(h^q) &= \mathcal{O}(h^p), \\
\mathcal{O}(h^p) \cdot \mathcal{O}(h^q) &= \mathcal{O}(h^{p+q}), \\
\mathcal{O}(h^p) - \mathcal{O}(h^p) &= \mathcal{O}(h^p), \\
\frac{1}{\mathcal{O}(h^p)} &\neq \mathcal{O}\left(\frac{1}{h^p}\right).
\end{aligned}$$

Exercise No. 4 – Resizing $\mathcal{O}$ (3×2=6 points)

Let the following functions be given, with $h \in \mathbb{R}$ *small*:

$$\begin{aligned} a(h) &:= h + h^2 + 10^{20}h^3 \\ b_1(h) &:= h + h^2 + 10^{20}h^3 + 10^{-100}h^4 \\ b_2(h) &:= -h - h^2 + 10^{20}h^3 + 10^{-100}h^4. \end{aligned}$$

Write down the results of the following operations in terms of $\mathcal{O}(h^k)$, $k \in \mathbb{N}$. Give each time a reason for your answer.

(a) $a(h) \cdot b_1(h)$

(b) $b_1(h) - a(h)$

(c) $b_1(h) + b_2(h)$