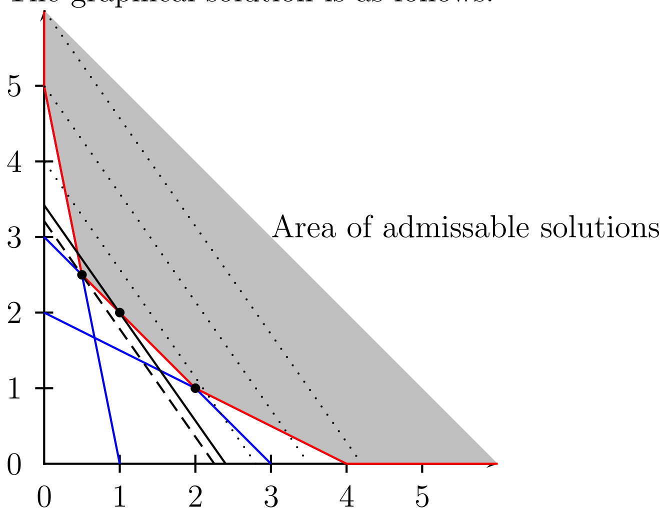
Example Solutions for Assignment 6

Problem 1 (Programming of First Order)

1. The linear program is given by

$$\begin{cases} 10x_1 + 7x_2 \to \min \\ 20x_1 + 20x_2 \geq 60 \\ 15x_1 + 3x_2 \geq 15 \\ 5x_1 + 10x_2 \geq 20 \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases}$$

2. The graphical solution is as follows:



At first, one formulates the linear constraints in terms of a linear function and draws this line into the graph (blue lines). The gray area denotes the area of admissible solutions. The black lines result for different resulting values of the minimisation condition $f(x, y) = 10x + 7y$. Each crossing with the cone can be considered as a candidate. The correct minimum can be found for $f(0.5, 2.5) = 22.5$. However, one may also argue that wafers should be intact for manifacturing, so we only consider integer values for x and y. For $f(1, 2) = 24$ we have found the best solution.

3. In general a linear optimisation problem can be considered with

$$\begin{cases} \text{minimize} & c_1x_1 + \ldots + c_nx_n = \vec{c}^\top \vec{x} \\ \text{such that} & A\vec{x} \leq b \\ & \vec{x} \geq 0 \end{cases}$$

In this notation, we have

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

for all $i = 1, \ldots, n$. By introducing helper variables $x_{n+1}, \ldots, x_{2n}$, we can rewrite this as

$$\sum_{j=1}^{n} a_{ij}x_j + x_{n+i} = b_i,$$

resulting in the alternate version of the program

$$\begin{cases} \text{minimise} & c_1x_1 + \ldots + c_nx_n = \vec{c}^\top \vec{x} \\ \text{such that} & A^*\vec{x}^* = b \\ & \vec{x}^* \geq 0 \end{cases}$$

Then the dual formulation is given as

$$\begin{cases} \text{maximise} & \vec{b}^\top \vec{y} \\ \text{such that} & A^\top \vec{y} \leq c \\ & \vec{x}^* \geq 0 \end{cases}$$

It is then simple to show weak duality $c^\top x \geq b^\top y$ by

$$c^\top x = x^\top c \geq x^\top (A^\top y) = x^\top A^\top y = (Ax)^\top y = b^\top y$$

Duality gap is also possible, for example if a problem has an unbounded solution and its dual an infeasible solution (or vice versa).

Problem 2 (Hanging out with Joseph Louis) The problem formulation is given by

$$\begin{cases} \text{minimise} & f(x_1, x_2) = -\frac{1}{2}\sqrt{x_1} - \frac{1}{2}x_2 \\ \text{such that} & x_1 \geq 0.1 \\ & x_2 \geq 0 \\ & x_1 + x_2 \leq 1 \end{cases}$$

Then the Lagrangian is given as

$$L := -\frac{1}{2}\sqrt{x_1} - \frac{1}{2}x_2 + \mu_1(0.1 - x_1) + \mu_2(-x_2) + \mu_3(x_1 + x_2 - 1)$$

satisfying the minimisation formulation

$$\begin{cases} \text{minimise} & f(x) & \\ \text{such that} & g_i(x) \leq 0, & i = 1, \ldots, m \\ & h_i(x) = 0, & j = 1, \ldots, p \end{cases}$$

With the settings

$$\nabla f(x) = \begin{pmatrix} -\frac{1}{4\sqrt{x_1}} \\ -\frac{1}{2} \end{pmatrix}, \qquad \nabla g_1(x) = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$
$$\nabla g_2(x) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \qquad \nabla g_3(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

we can derive the KKT conditions for this problem with

$$\begin{cases} \begin{pmatrix} -\frac{1}{4\sqrt{x_1}} \\ -\frac{1}{2} \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \mu_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} & = & 0 \\ \mu_1(0.1 - x_1) & = & 0 \\ \mu_2(-x_2) & = & 0 \\ \mu_3(x_1 + x_2 - 1) & = & 0 \\ \mu_1 & \geq & 0 \\ \mu_2 & \geq & 0 \\ \mu_3 & \geq & 0 \\ 0.1 - x_1 & \leq & 0 \\ -x_2 & \leq & 0 \\ x_1 + x_2 - 1 & \leq & 0 \end{cases}$$

With careful plugging in and calculations, one can compute a non-trivial solution for $\mu_1 = 0, \mu_2 = 0, \mu_3 = \frac{1}{2}, x_1 = \frac{1}{4}, x_2 = \frac{3}{4}$.

Problem 3 (Getting a fix)

Recall Banach's fixed point theorem: Let $X \subseteq \mathbb{R}^N$ be a non-empty and closed set and $P : X \to X$ a self-contracting mapping onto itself, i.e. there exists a constant $\alpha \in (0, 1)$ with

$$\|P(x) - P(y)\| \;\; \leq \;\; \alpha \|x - y\|$$

for all $x, y \in X$. Then P has exactly one fixed point x^* in X. Furthermore, with the iterative algorithm $x^{k+1} := P(x^k)$, $k = 0, 1, \ldots,$ and $x^* \in X$, each arbitrary sequence $\{x^k\}$ converges against x^*.

As the iterative algorithm is given as $P(x^j) := \lambda^{j+1}(x^j) = \lambda^j + r^j h(x^j)$, we can compute

$$\begin{aligned}
\|P(x^j) - P(y^j)\|^2 &= \|\lambda^j + r^j h(x^j) - \lambda^j - r^j h(y^j)\|^2 \\
&= \|r^j h(x^j) - r^j h(y^j)\|^2 \\
&= |r^j| \| \underbrace{h(x^j)}_{=0} - \underbrace{h(y^j)}_{=0} \| \\
&\leq |r^j| \|x^j - y^j\|^2 = c r^{j-1} \|x^j - y^j\|^2
\end{aligned}$$

and as $0 \leq r^j \leq 1$, all conditions for Banach are fulfilled.

Problem 4 (Operation KKT)

1. The KKT conditions for the given problem can be given with the help of the derivatives

$$\nabla f(x) = \begin{pmatrix} 3x_1^2 \\ 3x_2^2 \\ 3x_3^2 \end{pmatrix}, \quad \nabla g(x) = \begin{pmatrix} 2x_1 \\ 2x_2 \\ 3 \end{pmatrix}, \quad \nabla h(x) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \tag{1}$$

   as

$$\begin{cases} \begin{pmatrix} 3x_1^2 \\ 3x_2^2 \\ 3x_3^2 \end{pmatrix} + \mu \begin{pmatrix} 2x_1 \\ 2x_2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ \mu(x_1^2 + x_2^2 + 3x_3 + \frac{5}{2}) \\ \mu \geq 0 \\ x_1 + x_2 + x_3 + 2 = 0 \\ x_1^2 + x_2^2 + 3x_3 + \frac{5}{2} \leq 0 \end{cases} \tag{2}$$

2. The Lagrangian is given by

$$\begin{aligned}
L(x, \mu, \lambda) &= f(x) + \mu g(x) + \lambda h(x) \qquad (3) \\
&= x_1^3 + x_2^3 + x_3^3 + \mu(x_1^2 + x_2^2 + 3x_3 + \frac{5}{2}) + \lambda(x_1 + x_2 + x_3 + 2)
\end{aligned}$$

   and the corresponding dual function therefore is

$$\Theta(\vec{\mu}, \vec{\lambda}) = \inf_x x_1^3 + x_2^3 + x_3^3 + \mu(x_1^2 + x_2^2 + 3x_3 + \frac{5}{2}) + \lambda(x_1 + x_2 + x_3 + 2).$$

3. A simple look at the original function $f(x) = x_1^3 + x_2^3 + x_3^3$ shows immediately, that the function itself that should be minimized is not convex, hence Theorem 12.2 does not apply here. Furthermore, if one looks at definition of the duality gap, one can determine both sides of the difference. The right hand side $\max\{\Theta(\vec{\mu}, \vec{\lambda}) \mid \mu_i(x) \geq 0 \; \forall i, j\}$ can be evaluated internally (or at least the dual function that is computed) by means of methods such as Newton's method. An application reveals as a minimum $x_1 = -0.5, x_2 = -0.5, x_3 = -1$. However, an optimal value for $\mu = -0.5625$ is negative and clearly violates the conditions. However, plugged into the function $f(x)$ a minimum would be found with $f(x) = -\frac{10}{8}$. However, considering now also the left hand side of the duality gap formulation, i.e. $\min\{f(x) \mid g_i(x) \leq 0, \;\; h_j(x) = 0 \; \forall i, j\}$, we can have a look at the conditions in it first. For example, the condition $x_1^2 + x_2^2 + 3x_3 \leq \frac{-5}{2}$ leads to the conclusion, that in order to fulfill this condition, x_1 and x_2 should be close to zero, such that $x_3 \leq -\frac{5}{6}$. The second condition $x_1 + x_2 + x_3 + 2 = 0$ however shows that for a suggested $x_1 = x_2 = 0, \; x_3 = -2$, which works according to the previous statement. However, inserted into the initial function gives $f(x) = -8$, in contrast to the other result. It should be noted, that one can repeat this exercise with the program

$$\text{minimize} \qquad x_1^2 + x_2^2 + x_3^2 \tag{4}$$

subject to

$$x_1^2 + x_2^2 + 3x_3 \;\; \leq \;\; -\frac{5}{2} \qquad \text{and} \tag{5}$$

$$x_1 + x_2 + x_3 = -2 \tag{6}$$

as a further exercise on that problem.
