Numerical Algorithms for Visual Computing III 2011
Example Solutions for Assignment 5

Problem 1 (The Convex Complexity)

1. To be proven: Choose f differentiable. Then f is convex if and only if S is convex and

$$f(y) \quad \geq \quad f(x) + (\nabla f(x))^\top (y - x)$$

for all $x, y \in S$.

Proof:
„$\Rightarrow$“Let $S \subseteq \mathbb{R}^N$. We prove first for $N = 1$ and later generalize. Assume f convex and $x, y \in S$. As S is convex, it follows that there exists a t with $0 < t \leq 1$, such that $x + t(y - x) \in S$ and by convexity:

$$\begin{aligned}
& f(x + t(y - x)) && \leq (1 - t)f(x) + tf(y) \\
\Leftrightarrow \quad & f(x + t(y - x)) - (1 - t)f(x) && \leq tf(y) \\
\Leftrightarrow \quad & f(y) && \geq f(x) + \frac{f(x + t(y - x)) - f(x)}{t} \\
\overset{t \to 0}{\Rightarrow} \quad & f(y) && \geq f(x) + f'(x)(y - x).
\end{aligned}$$

„$\Leftarrow$“Assume $f(y) \geq f(x) + f'(x)(y - x)$ hold for all $x, y \in S$. Let $x \neq y$, $0 \leq \Theta \leq 1$ and $z = \Theta x + (1 - \Theta)y$. From this we can compute two inequalitites

$$\begin{aligned}
f(x) \quad & \geq \quad f(z) + f'(z)(x - z) \\
f(y) \quad & \geq \quad f(z) + f'(z)(y - z)
\end{aligned}$$

Adding and weighting both equations with Θ and $(1 - \Theta)$ respectively gives

$$\begin{aligned}
& \Theta f(x) + (1 - \Theta)f(y) && \geq \Theta(f(z) + f'(z)(x - z)) + (1 - \Theta)f(z) + f'(z)(y - z) \\
\Leftrightarrow \quad & \Theta f(x) + (1 - \Theta)f(y) && f(z) + f'(z)(\Theta(x - z) + (1 - \Theta)(y - z)) \\
\Leftrightarrow \quad & \Theta f(x) + (1 - \Theta)f(y) && f(z),
\end{aligned}$$

i.e. f is convex.

General proof: Let $f : \mathbb{R}^N \to \mathbb{R}^N$, and $x, y \in \mathbb{R}^N$
„$\Rightarrow$“ f bounded by $g(t) = f(ty + (1-t)x)$ with a gradient $g'(t) = (\nabla f(ty + (1-t)y))^\top (y-x)$. Then if f convex, g is also convex. If one sets $t = 1$, then

$$\begin{aligned} & g(1) && \geq g(0) + g'(0) \\ \Rightarrow\; & f(y) && \geq f(x) + (\nabla f(x))^\top (y-x). \end{aligned}$$

„$\Leftarrow$“Assume $f(y) \geq f(x) + (\nabla f(x))^\top (y-x)$ holds for any x, y, so if $ty + (1-t)x \in S$ and $\bar{t}y + (1-\bar{t}) \in S$, we have

$$f(ty + (1-t)x) \quad \geq \quad f(\bar{t}y + (1-\bar{t})) + (\nabla(\bar{t}y + (1-\bar{t})))^\top (x-y)(t-\bar{t}),$$

i.e. $g(t) \geq g(\bar{t})(t - \bar{t})$, and from this g is convex.

2. Let $f_1, \dots, f_n$ be convex. Let $0 \leq \Theta \leq 1$, $x, y \in S$. Then

$$\begin{aligned} f(\Theta x + (1-\Theta)y) &= \max\{f_1(\Theta x + (1-\Theta)y), \dots, f_n(\Theta x + (1-\Theta)y)\} \\ &\leq \max\{\Theta f_1(x) + (1-\Theta)f_1(y), \dots, \Theta f_n(x) + (1-\Theta)f_n(y)\} \\ &\leq \Theta \max\{f_1(x), \dots, f_n(x)\} + (1-\Theta)\max\{f_1(x), \dots, f_n(x)\} \\ &= \Theta f(x) + (1-\Theta)f(y) \end{aligned}$$

3. Given the function $f(x,y) = \frac{x^2}{y}$, we can compute

$$\begin{aligned} \nabla f(x,y) &= \begin{pmatrix} \frac{2x}{y} \\ \frac{-x^2}{y} \end{pmatrix} \\ Hf(x,y) &= \frac{2}{y^3} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} \end{aligned}$$

From this it is easy to show that f is convex for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}, y \geq 0$.

Problem 2 (The Ellipsoid Condition)

1. Try to use a given matrix, e.g.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

and plug in some values in both notations.

2. At first, let us recapitulate $\mathcal{E} = \{x \mid (x - x_c)^\top P^{-1}(x - x_c)\} = \{x_c + Aq \mid \|q\| \leq 1\}$. Plugging into the formulae from the lecture, we get

$$\begin{aligned}\sup_{z\in\mathcal{E}} q^\top z - \inf_{z\in\mathcal{E}} q^\top z &= q^\top(x_c + Aq) - q^\top(x_c - Aq)\\ &= q^\top Aq + q^\top x_c - q^\top x_c + q^\top Aq = 2q^\top Aq = 2\|A^{\frac{1}{2}}q\|_2\end{aligned}$$

From this, we can see that $W_{min} = 2\lambda_{min}(A)^{\frac{1}{2}}$ and $W_{max} = 2\lambda_{max}(A)^{\frac{1}{2}}$ and the condition number $\text{cond}(A) = \frac{\lambda_{max}}{\lambda_{min}} = \kappa(A)$.

Problem 3 (The Himmelblau Sky)

General stuff:

$$\begin{aligned}f(x,y) &= (x^2 + y - 11)^2 + (x + y^2 - 7)^2\\ \nabla f(x,y) &= \begin{pmatrix} 4x^3 + 4xy - 42x + 2y^2 - 14 \\ 2x^2 + 4xy + 4y^3 - 26y - 22 \end{pmatrix}\\ Hf(x,y) &= \begin{pmatrix} 12x^2 + 4y - 42 & 4(x+y) \\ 4(x+y) & 4x + 12y^2 - 26 \end{pmatrix}\end{aligned}$$

Evaluated at point $(2,3)^\top$:

$$\begin{aligned}f(2,3) &= 32\\ \nabla f(2,3) &= \begin{pmatrix} -24 \\ 40 \end{pmatrix}\\ Hf(x,y) &= \begin{pmatrix} 18 & 20 \\ 20 & 90 \end{pmatrix}\end{aligned}$$

Taylor-Expansion:

$$\begin{aligned}T_2 f(x,y) &= f((2,3)^\top) + \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)^\top \nabla f((2,3)^\top)\\ &+ \frac{1}{2}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)^\top \begin{pmatrix} 18 & 20 \\ 20 & 90 \end{pmatrix}\left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \end{pmatrix}\right)\\ &= 9x^2 + 20xy + 45y^2 - 120x + 521 - 270y\end{aligned}$$

Problem 4 (The Conal Obfuscation)

1. Cone or Convex Cone?

- $\{x \in \mathbb{R}^N \mid x \geq 0\}$ convex cone
- $\{x = (x_1, x_2)^\top \in \mathbb{R}^2 \mid x_1 \geq 0 \wedge x_2 = 0 \ \vee \ x_1 = 0 \wedge x_2 \geq 0\}$ cone, but not convex
- $\{x \in \mathbb{R}^N \mid x \geq 0\} \cup \{x \in \mathbb{R}^N \mid x \leq 0\}$ cone, but not convex

2. To be shown: A point $y_x \in S$ is a projection $p_C(x)$ of $x \in \mathbb{R}^N$ onto C if and only if $\langle x - y_x, y - y_x\rangle \leq 0 \ \forall y \in C$.
„$\Rightarrow$ "Assume y_x is the solution of $\inf\{\frac{1}{2}\|y-x\|^2, y \in C\}$ w.r.t. x. Let $y \in C$ such that $y_x + \alpha(y - y_x) \in C$ for any $\alpha \in (0,1)$. Then with $f_x := \frac{1}{2}\|y-x\|^2$

$$\begin{aligned} f_x(y_x) &\leq f_x(y_x + \alpha(y - y_x)) \\ &= \frac{1}{2}\|y_x - x + \alpha(y - y_x)\|^2 \\ &= \underbrace{\frac{1}{2}\|y_x - x\|^2}_{\geq 0} + \alpha\langle y_x - x, y - y_x\rangle + \underbrace{\frac{1}{2}\alpha^2\|y - y_x\|^2}_{\geq 0} \\ \Rightarrow 0 &\geq \alpha\langle y_x - x, y - y_x\rangle + \alpha^2\|y - y_x\|^2 \\ \overset{\alpha>0}{\Rightarrow} 0 &\leq \langle y_x - x, y - y_x\rangle + \alpha\|y - y_x\|^2 \\ \overset{\alpha>0}{\Rightarrow} 0 &\leq \langle y_x - x, y - y_x\rangle \\ \Leftrightarrow 0 &\geq \langle x - y_x, y - y_x\rangle \end{aligned}$$

„$\Leftarrow$ "Let $y_x \in C$ with $\langle x - y_x, y - y_x\rangle \leq 0$ for all $y \in C$. If $y_x = x$ then y_x solves trivially $\inf\{\frac{1}{2}\|y-x\|^2, y \in C\}$. Else $y_x \neq x$, so we write for $y \in C$

$$\begin{aligned} 0 &\geq \langle x - y_x, y - y_x\rangle \\ &= \langle x - y_x, y - x + x - y_x\rangle \\ &= \|x - y_x\|^2 + \langle x - y_x, y - x\rangle \\ &\geq \|x - y_x\|^2 - \|x - y\| \cdot \|x - y_x\| \end{aligned}$$

by means of the Cauchy-Schwartz inequality. Dividing by $\|x - y_x\| > 0$ gives

$$\begin{aligned} 0 &\geq \|x - y_x\| - \|x - y\| \\ \Rightarrow \ \|x - y\| &\geq \|x - y_x\| \end{aligned}$$

for all $y \in C$. This means that y_x solves $\inf\{\frac{1}{2}\|y-x\|^2$.

3. Let us recount the definition of a scalar product, i.e. $\langle v, w\rangle = \cos\Theta\|v\| \cdot \|w\|$ for all $v, w \in \mathbb{R}^N$. Then the scalar product $\langle x - y_x, y - y_x\rangle \leq 0$ means

that the angle between $y - y_x$ and $x - y_x$ is obtuse („stumpfer Winkel“) for any $y \in C$. This is equivalent to $\langle x - p_C(x), y\rangle \leq \langle x - p_C(x), p_C(x)\rangle$ for all $y \in C$, i.e. p_C lies in the surface of C exposed by $x - p_C(x)$.

4. Let K be a closed convex cone. It is to be shown that $y_x = p_K(x)$ if and only if $\langle x - y_x, y_x\rangle = 0,\ y_x \in K,\ x - y_x \in K^p$.
„$\Rightarrow$“$y_x = p_K(x)$ satisfies $\langle x - y_x, y - y_x\rangle \leq 0$ for all $y \in K$ (see exercise part 4b). Use $y = \alpha y_x$ with $\alpha \geq 0$, this implies

$$(\alpha - 1)\langle x - y_x, y_x\rangle \ \leq \ 0 \qquad \forall \alpha \geq 0.$$

Since $(\alpha - 1) \in [-1, \infty)$ this implies

$$\langle x - y_x, y_x\rangle \ = \ 0$$

and $\langle x - y_x, y - y_x\rangle \leq 0$ becomes

$$\begin{aligned} &\langle y, x - y_x\rangle = 0 \qquad \forall y \in K \\ \Leftrightarrow\ & x - y_x \in K^p \end{aligned}$$

with K^p being the polar cone.

„$\Leftarrow$“: Let y_x satisfy $\langle x - y_x, y_x\rangle = 0$ for any $y \in K$. Then

$$f_x(y) = \frac{1}{2}\|x - y_x + y_x - y\|^2 \ \geq \ f_x(y_x) + \langle x - y_x, y_x - y\rangle$$

but with $\langle x - y_x, y_x\rangle = 0$:

$$\begin{aligned} & \langle x - y_x, y_x - y\rangle \\ = \ & \underbrace{\langle x - y_x, y_x\rangle}_{=0} - \langle x - y_x, y\rangle \\ = \ & -\langle x - y_x, y\rangle \geq 0, \end{aligned}$$

hence $f_x(y) \geq f_x(y_x)$ and y_x solves $\inf\{\frac{1}{2}\|y - x\|^2, y \in C\}$.
