Example Solutions for Assignment 1

Problem 1 (Germany's Next Curve Model)

1. Similar to the example in the lecture, we can compute

$$
\begin{aligned}
&\frac{1}{2}\int_0^R \|C'(x)\|_2^2\,\mathrm{d}x \\
= &\frac{1}{2}\int_0^R \left\| \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x}[1] \\ \frac{\mathrm{d}}{\mathrm{d}x}[u(x)] \end{pmatrix} \right\|_2^2 \mathrm{d}x \\
= &\frac{1}{2}\int_0^R \left\| \begin{pmatrix} 0 \\ u'(x) \end{pmatrix} \right\|_2^2 \mathrm{d}x = \frac{1}{2}\int_0^R |u'(x)|^2 \mathrm{d}x = \frac{1}{2}\int_0^R (u'(x))^2 \mathrm{d}x.
\end{aligned}
$$

2. At first, the Lagrangian is given as

$$F(x, u(x), u'(x)) \quad = \quad (u'(x))^2$$

Then, we can compute the Euler-Lagrange equation

$$F_u - \frac{\mathrm{d}}{\mathrm{d}x}F_{u'} \quad = \quad 0$$

with

$$
\begin{aligned}
F_u &= 0 \\
F_{u'} &= u'(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}F_{u'} &= u''(x)
\end{aligned}
$$

resulting in the Euler-Lagrange equation

$$u''(x) \quad = \quad 0.$$

This is also called the Laplace equation, which is the steady-state solution of the 1-D heat equation

$$u_t \quad = \quad u_{xx}$$

Problem 2 (Optimal Prime)

1. General approach: Recall from the lectures of Mathematics for Computer Scientists that a function in several variables is convex if and only if its Hessian matrix is positive semidefinite. Given the Lagrangian in the form

$$F(x, \lambda, \eta) \;=\; \lambda\sqrt{1+\eta^2}$$

Then the Hessian is given as

$$\mathcal{H}F \;=\; \begin{pmatrix} f_{\eta\eta} & f_{\eta\lambda} \\ f_{\eta\lambda} & f_{\lambda\lambda} \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{\sqrt{1+\eta^2}} - \frac{\lambda\eta}{(1+\eta^2)^{\frac{3}{2}}} & \frac{\lambda\eta}{\sqrt{1+\eta^2}} \\ \frac{\lambda\eta}{\sqrt{1+\eta^2}} & 0 \end{pmatrix}.$$

One method to discover positive definiteness of a matrix is to consider the quadratic form of a matrix, by considering

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \mathcal{H}F \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;\geq\; 0.$$

After some calculations, we see that

$$x_1^2\left(\frac{\lambda}{\sqrt{1+\eta^2}} - \frac{\lambda\eta}{(1+\eta^2)^{\frac{3}{2}}}\right) + 2x_1x_2\left(\frac{\lambda\eta}{\sqrt{1+\eta^2}}\right).$$

Obviously, if $\lambda < 0$, then the result is no longer positive. Furthermore it is very difficult to determine if the remaining terms can compensate this. Now if we consider the alternate definition, this simplifies the convexity property drastically. The integrand function is now

$$F(\lambda, \eta) \;=\; \lambda\sqrt{1+\eta^2}.$$

If we fix η, i.e. we are setting this variable to a constant value, then this function is convex for all $\lambda \geq 0$. If $\lambda > 0$ fixed, then η is strictly convex. For this example, the alternative theorem is more simple.

2. For this simple case the Euler-Lagrange equation also simplifies, which results in

$$\frac{\mathrm{d}}{\mathrm{d}x}[F(u(x), u'(x)) - u'(x)F_\eta(u(x), u'(x)))] \;=\; 0.$$

With

$$F_\eta \;=\; \frac{\lambda\eta}{\sqrt{1+\eta^2}}$$

the resulting Euler-Lagrange equation is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}x}[\lambda\sqrt{1+\eta^2} - \eta\frac{\lambda\eta}{\sqrt{1+\eta^2}}] = 0.$$

Integrating over the domain gives us then

$$\lambda\sqrt{1+\eta^2} - \eta\frac{\lambda\eta}{\sqrt{1+\eta^2}} = c.$$

Problem 3 (Condition Zero) Let us first consider the Euler-Lagrange equation of the given energy functional. The Lagrangian is given as

$$F(x,\lambda,\eta) = \eta^2 + (\lambda-2)^2.$$

This, together with

$$\begin{aligned} F_\lambda &= 2(\lambda-2) \\ F_\eta &= \eta \end{aligned}$$

leads to the E-L equation

$$u(x) - 2 = u''(x)$$

It is now our task to find a fitting function for the given constraints.

1. Is the function uniquely determinable at hand of the given constraints? No, because some of the constraints are quite arbitrary, i.e. $2 \leq u(0) \leq 3$. Furthermore, some constraints are mutual exclusive, e.g. $u(x) = 2 + 0.5\exp(x)$ has the property of fulfilling $2 \leq u(0) \leq 3$ but not $u(\log 2) = 1$. One needs additional clearer constraints for a suitable solution. Additionally, one condition is also given by the initial constraints, as $u''(\log 2) = u(\log 2) - 2 = 1 - 2 = -1$.

2. If we consider $u'(0) = 0$, then we have to find a function that fits that constraint. One possible family of functions for the second derivative is $u''(x) = m_1\exp(x) + m_2\exp(-x)$. However, starting from u"(x), we get two additional variables a and b from the integration. Hence we only consider for a unique solution $m = m_1 = m_2$. To this end, we can plug our constraint $u''(\log 2) = -1$ into this candidate equation, giving us $m = -\frac{2}{5}$. A proper primitive function is then given

as $u'(x) = a - \frac{2}{5}(\exp(x) - \exp(-x))$. With $u'(0) = 0$, this gives $u'(0) = a$ which gives $a = 0$. Then we integrate further, giving us $u(x) = b - \frac{2}{5}(\exp(x) + \exp(-x))$. With the two constrainst given, this becomes a linear system of equations, which results in $u(x) = 2 - \frac{2}{5}(\exp(x) + \exp(-x))$. Unfortunately the solution for $u(0)$ is $\frac{6}{5}$, so this also violates the $2 \leq u(0) \leq 3$-condition.

3. Let us recapitulate: We know from part (a) that we need to solve the differential equation $u(x) - 2 = u''(x)$ together with the side constraint $u(\log 2) = 1$. From this we can compute $u''(\log 2) = -1$. Another constraint that we are getting is $u(0) = 2$, which results in $u''(0) = 0$. From this we have 4 constraints and four unknowns. Let us consider the candidate function for the second derivative $f''(x) = m_1 \exp(x) + m_2 \exp(-x)$. A proper primitive function would be $f(x) = m_1 \exp(x) + m_2 \exp(-x) + ax + b$. Let us now plug in the constraints for the second derivative: $u''(0) = m_1 + m_2 \stackrel{!}{=} 0$, i.e. $m_1 = -m_2$. Then with $u''(\log 2) = 2m_1 - 0.5m_1 \stackrel{!}{=} -1$, i.e. $m_1 = -\frac{2}{3}$. Plugging this into the candidate function of $f(x)$, we get $f(x) = -\frac{2}{3}\exp(x) + \frac{2}{3}\exp(-x) + ax + b$. Plugging now the given constraint $u(0) = 2$ gives $b = 2$ and $u(\log 2) = 2$ gives $a = 0$, hence the function that correctly satisfies all constraints is the function $f(x) = 2 - \frac{2}{3}(\exp(x) - \exp(-x))$.

Problem 4 (Channel Reloaded) Assuming $u(x) > 0$, we can compute the Lagrangian

$$F(x, \lambda, \eta) = \sqrt{1 + \eta^2} + z\lambda.$$

This function is convex in (λ, η), guaranteeing uniqueness of the optimal solution. With

$$\begin{aligned} F_\lambda &= z \\ F_\eta &= \frac{\eta}{\sqrt{1 + \eta^2}} \end{aligned}$$

we arrive at the Euler-Lagrange equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\eta}{\sqrt{1 + \eta^2}}\right) &= z \\ \left(\frac{\eta}{\sqrt{1 + \eta^2}}\right) &= zx + c \end{aligned}$$

After some computations, this can be simplified to

$$u'(x) = \frac{zx + c}{\sqrt{1 - \sqrt{zx + c)^2}}}$$

Then, we have to compute the anti-derivative, resulting in

$$\begin{aligned} u(x) &= \int_0^x \frac{zs + c}{\sqrt{1 - \sqrt{zs + c)^2}}} \mathrm{d}s \\ &= \frac{1}{z}(\sqrt{1 - c^2} - \sqrt{1 - (zx + c)^2})). \end{aligned}$$

Introducing the condition $u(1) = 0$ leads to $c = -\frac{z}{2}$ and hence

$$\frac{1}{2z}(\sqrt{4 - z^2} - \sqrt{4 - z^2(2x - 1)^2}).$$

This optimum solution describes an arc of a circle.
