Numerical Algorithms for Visual Computing III: Optimisation

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Assignment 5

Exercise No. 1 – The Convex Complexity ($5 + 3 + 3 + 3 = 14$ points)

(a) Prove the following theorem from the lecture:
Suppose $f \in \mathcal{C}^1$. Then f is convex on a set S if and only if

$$f(y) \quad \geq \quad f(x) + (\nabla f(x))^\top (y - x)$$

for all $x, y \in S$.

(b) Given convex functions $f_1, \dots, f_m$. Then the pointwise maximum $f(x) = \max\{f_1(x), \dots, f_m(x)\}$ is also convex.

(c) Prove that the quadratic-over-linear function $f(x, y) = \frac{x^2}{y}$ is convex.

(d) Prove that the sublevel sets of a convex function are convex. Is the converse direction valid?

Hint: Make use of Jensen's inequality: A function $f : \mathbb{R}^N \to \mathbb{R}$ is convex if $S \in \mathbb{R}^N$ is a convex set and if for all $x, y \in \mathbb{R}^N$ and $\Theta \in [0, 1]$ it holds

$$f(\Theta x + (1 - \Theta)y) \quad \leq \quad \Theta f(x) + (1 - \Theta) f(y).$$

Exercise No. 2 – The Ellipsoid Condition ($1 + 3 = 4$ points)

1. Given a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, there are several ways to write an ellipsoid set:

   - $\mathcal{E} = \{x \mid (x - x_c)^\top P^{-1} (x - x_c) \leq 1\}$
   - $\mathcal{E} = \{x_c + Au \mid \|u\| \leq 1\}$

   with $A = P^{\frac{1}{2}}$. Show (e.g. by comparible sketches) that both definitions describe the same ellipsoid.

2. Compute the condition number of an ellipsoid.

Exercise No. 3 – The Himmelblau Sky (2 points)

Given the Himmelblau function

$$f(x, y) \quad = \quad (x^2 + y - 11)^2 + (x + y^2 - 7)^2 \tag{1}$$

Perform a Taylor expansion up to second order around the point $(2, 3)^\top$.

Exercise No. 4 – The Conal Obfuscation ($3 + 5 + 2 + 5^* = 15$ points)

In this exercise we want to consider the notion of projection and convex cones in the context of optimisation.
Cones: A set $K \subseteq \mathbb{R}^N$ is a *cone* if $x \in K \Rightarrow \{\alpha x \;\mid\; \alpha > 0\}$. A cone $K \subset \mathbb{R}^N$ is convex if and only if $K + K \subseteq K$. The polar cone K^p is defined as $K^p := \{s \in \mathbb{R}^N \;\mid\; \langle s, x\rangle \leq 0 \text{ for all } x \in K\}$.
Projection: Consider the minimisation problem

$$\inf\left\{\underbrace{\frac{1}{2}\|y-x\|^2}_{=:f_x(y)}, y \in C\right\}$$

for K nonempty, closed, convex set. Then considering the level set for $k \in C$

$$S := \{y \in \mathbb{R}^N : f_x(y) \leq f_x(c)\}$$

the minimisation process becomes

$$\inf\{f_x(y) : \; y \in C \cup S, y \in C\}$$

For a convex function we can define an operator $p_c : C \to C$ that assigns to each $x \in C$ the unique solution of the minimisation problem.

1. Are the following sets cones or even convex cones?
   - $\{x \in \mathbb{R}^N \;\mid\; x \geq 0\}$
   - $\{x = (x_1, x_2)^\top \in \mathbb{R}^2 \;\mid\; x_1 \geq 0 \wedge x_2 = 0 \;\vee\; x_1 = 0 \wedge x_2 \geq 0\}$
   - $\{x \in \mathbb{R}^N \;\mid\; x \geq 0\} \cup \{x \in \mathbb{R}^N \;\mid\; x \leq 0\}$
2. Prove: A point $y_x \in C$ is the projection is the projection $p_c(x)$ of $x \in \mathbb{R}^N$ if and only if

$$\langle x - y_x, y - y_x\rangle \quad \leq \quad 0 \qquad \text{for all } x, y \in C$$

3. What is the geometrical interpretation of part (b)?
4. Prove: Let K be a closed convex cone. Then $y_x = p_K(x)$ if and only if

$$\langle x - y_x, y_x\rangle \quad = \quad 0, \qquad y_x \in K, \qquad x - y_x \in K^p.$$