Numerical Algorithms for Visual Computing III: Optimisation

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Assignment 4 – Practical Assignments

Exercise No. 1 – Comparison of Gradient Descent methods (3+3+3=9 points)

Consider the following already discretised 1-D energy functional:

$$E_f(u) \quad = \quad \frac{1}{2}\sum_{k=1}^{N}(u_k - f_k)^2 + \frac{\alpha}{2}\sum_{l\in\mathcal{N}(k)}(\frac{u_k - u_l}{\Delta x})^2, \tag{1}$$

with $\mathcal{N}(k)$ being the neighborhood around pixel u_k together with the signal

$$f(k) \quad = \quad 20 * sin(k) + 20 \qquad k = 0, \ldots, 19. \tag{2}$$

(a) Compute the gradient $\nabla E_f(u)$ and the Hessian matrix $HE_f(u)$.

(b) Implement the method of 'Gradient Descent Line Search' (GDLS) for the minimisation of $E_f(u)$ with the use of the backtracking line search algorithm. What is an appropriate stopping criterion for the algorithm? Is your result an optimal solution for the minimisation problem? Discuss.

(c) Implement Newton's method (without stepsize control) for the minimisation of $E_f(u)$. What differences occur in comparison to the GDLS algorithm? Do you get the same results? Elaborate.

Exercise No. 2 – Playing with the banana (4+4+4=12 points)

For testing optimisation algorithms, there exist several testing functions to test speed and convergence of the algorithms. We are going to have a look at two such functions. The first is Rosenbrock's banana function

$$f(x, y) \quad = \quad (1 - x)^2 + 100(y - x^2)^2,$$

the second is the Bazaraa-Shetty function

$$f(x, y) \quad = \quad (x - 2)^4 + (x - 2y)^2.$$

All three functions have each a global minimum .For that task, you should implement and compare the results of

(a) Gradient Descent with backtracking line search (GDLS)

(b) Newton's method without step size control

(c) Trust Region method (also give a good scaling strategy for the choice of α)

with each other concerning speed and proximity to the correct solution. You may use as a starting point for each problem the position $(-1.2, 1)^\top$.

Exercise No. 3 – Descending Into The Abyss (2+2+2+3=9 points)

(a) Given is the 1D-problem $f(x) = x^2$. Try to find the global minimum with a gradient descend method and a fixed step size $\gamma = 1$. Explain your results!

(b) Given is the 1D-problem $f(x) = x^4 + 3x^3 - 3x^2 - 7x + 6$. Determine the extreme points analytically. Implement Newton's method without stepsize control for that problem and use different starting points for your algorithm. Does the result always end up in a global minimum?

(c) Given is the 1D-problem $f(x) = ax - \ln(x)$ for an arbitrary constant a. Devise an iterative scheme based on Newton's method. What is the global minimum of the function? Is this global minimum reachable for all starting points?

(d) Given the function $f(x) = x \arctan(x) - \frac{1}{2}\log(x^2 + 1)$. Try to find the Minimum of the function by use of Newton's method with varying starting points 1 and 1.5. What is your result? Can you elaborate on the problem? To this end, try to consider analysing $f'(x) = \arctan(x)$ and sketch the 3 general possible gradient directions!

Exercise No. 4 – Watch your step! (10 extra points)

In the lecture we have only given a simple method for choosing the time step size. In the following we will consider one step size method by Armijo. The idea of Armijos method is to compute a step size σ_A such that for a given $\beta \in]0, 1[$ the following conditions hold:

$$f(x + \sigma_A d) \leq f(x) + \beta\sigma_A(\nabla f(x))^\top d \tag{3}$$

$$\sigma_a \geq -c\frac{(\nabla f(x))^\top d}{\|d\|^2} \tag{4}$$

with d being a search direction, c a constant and $(\nabla f(x))^\top d < 0$. Then the Armijo method is as follows: Given constants $0 < \sigma < 1$ $0 < \gamma$ and $0 < \beta_1 \leq \beta_2 < 1$.

1. Choose a starting step size $\sigma_0 \geq -\gamma\frac{(\nabla f(x))^\top d}{\|d\|^2}$ and set $j = 0$.
2. If condition (3) is fulfilled, i.e. $f(x + \sigma_J d) \leq f(x) + \beta\sigma_j(\nabla f(x))^\top d$ then set $\sigma_j = \sigma_A$ and stop.
3. Else choose $\sigma_{j+1} \in [\beta_1\sigma_j, \beta_2\sigma_j]$.
4. $j := j + 1$ and go to 2.

Extend your implementation for Newton's method from exercise 1 for the banana function and elaborate on the results.