Numerical Algorithms for Visual Computing III: Optimisation

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Released: 06.05.2011
Assigned to: Tutorial at 12.05.2009

Assignment 2 – Variations of Two Problems

Exercise No. 1 – Tortellini and Ambrosia (20 points)

We proudly introduce in this exercise the *Tortellini-Ambrosia (TA) model* for 1-D signal segmentation.

It is constituted as to find a minimiser (u, v) of the energy functional

$$E_{TA}(u,v) = \int_a^b \beta\,(u-f)^2 + v^2\,(u')^2 + \alpha\left(\gamma\,(v')^2 + \frac{(1-v)^2}{4\gamma}\right)\,dx$$

where

- u is the simplified signal
- v is a jump detector with $v \approx 0$ at jumps and $v \approx 1$ in smooth signal regions
- α, β and γ are positive, real-valued parameters

The tasks are as follows:

1. Compute the Euler-Lagrange equation for this model. **(2 pts)**
2. Set $w := (u, v)^\top$ and compute $\nabla_w E_{TA}$. **(2 pts)**
3. Is there a unique minimiser of the Tortellini-Ambrosia model? Elaborate on this question. **(4 pts)**
   Hint: Consider the E-L equation from part (1). Neglect all derivatives for v and try to construct a function $v(|u_x|)$. Plug this into a modified version of $E_{TA}(u, v)$ and consider the resulting regulariser!
4. Discretise the Euler-Lagrange equation(s) using a spatial grid with mesh width h. Also, make use of

   $$\phi''(x_j) \approx \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{h^2}$$

   if applicable for some function $\phi \in \{u, v\}$. Write down your discretisation. **(2 pts)**
5. Write down an iterative scheme for solving the Euler-Lagrange equations. **(2 pts)**
6. Discretise E_{TA} and compute the necessary condition for a minimiser. **(6 pts)**
7. Write down an iterative scheme for solving the latter system. **(2 pts)**

Exercise No. 2 – The Problem of Nail and Grandsonman$^{\text{TM}}$ **($6 + 4 = 10$ points)**

Let us consider an image sequence $f(x, y, t)$, where (x, y) denotes a pixel position in the image domain $\Omega \in \mathbb{R}^2$ and a time parameter $t \in [0, T]$. The goal is to find a optic flow field $(u(x, y, t), v(x, y, t))$ that describes a displacement vector between two subsequent frames t and $t + 1$. In the following, we will abbreviate $u \equiv u(x, y, t)$ and $v \equiv v(x, y, t)$ respectively. Furthermore, we consider the assumption that the brightness of pixels do not change between two pixels, i.e. the so-called greyvalue constancy assumption, which is given as

$$f(x, y, t) - f(x + u, y + v, t + 1) \quad \stackrel{!}{=} \quad 0.$$

By performing a first-order Taylor expansion and neglecting higher order terms, we arrive at the linearised optic flow constraint

$$f_x u + f_y v + f_t \quad \stackrel{!}{=} \quad 0$$

Furthermore, we want to minimise deviations from this constraint with an additional smoothness assumption. From this we can model a variational description of the optic flow problem as

$$E(u, v) \quad = \quad \int_\Omega ((f_x u + f_y v + f_t)^2 + \alpha V(\nabla f, \nabla u, \nabla v)) \, \mathrm{d}x \, \mathrm{d}y$$

with $V(\nabla f, \nabla u, \nabla v)$ being a regulariser and $\alpha > 1$ a regularisation parameter.
Nagel proposed a regulariser of the form

$$V(\nabla f, \nabla u, \nabla v) \quad := \quad \nabla u^\top D(\nabla f) \nabla u + \nabla v^\top D(\nabla f) \nabla v$$

with $D(\nabla f)$ being a regularised projection matrix on $\nabla f^\perp = (f_y, -f_x)^\top$ defined as

$$D(\nabla f) \quad := \quad \frac{1}{|\nabla f|^2 + 2\lambda^2} (\nabla f^\perp \nabla f^{\perp\top} + \lambda^2 f)$$

and λ being a contrast parameter.

1. Derive the Euler-Lagrange equation with Nagel's regulariser!
2. What happens if $D(\nabla f)$ is the identity matrix I?