Numerical Algorithms for Visual Computing III

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Assignment 1 (4 Problems)

Exercise No. 1 – Germany's Next Curve Model (3+3=6 points)

Given the following assumption: We want to compute the length of the curve given by by $C = (1, u(x))$ by means of

$$\int_0^R \frac{1}{2} \|C'(x)\|_2^2 \mathrm{d}x \tag{1}$$

1. Derive the minimization problem.

2. Derive the Euler-Lagrange equation. What PDE is resulting from this Euler-Lagrange equation?

Exercise No. 2 – Optimal Prime (4+4=8 points)

Consider the 1-D minimization problem of finding the function $u(x)$ which is the infimum of

$$I_1(u) \quad = \quad \int_a^b u(x)\sqrt{1 + (u'(x))^2}\, \mathrm{d}x \tag{2}$$

where $(a, b) \subset \mathbb{R}$.

1. Can you determine by Theorem 1-1, if an optimal solution is unique?
   Is the answer to the latter question different, or not, if you consider the following theorem:
   Theorem. *If u is the unique of (E-L) and F is convex with respect to the variable η for each fixed $x \in \Omega$ and $\lambda \in \mathbb{R}$, then u is also an optimal solution of the variational problem.*

2. It can be shown that in distinct cases where $F = F(\lambda, \eta)$, i.e. F is solely depending on λ and η for a fixed $x \in \Omega$, that the Euler-Lagrange equation simplifies to

   $$\frac{\mathrm{d}}{\mathrm{d}x}[F(u(x), u'(x)) - u'(x)F_\eta(u(x), u'(x)))] \quad = \quad 0. \tag{3}$$

   Compute the Euler-Lagrange equation for this special case for the minimization problem given by (2).

Exercise No. 3 – Condition Zero (2+3+3=8 points)

Given the following 1-D energy functional

$$I_2(u) \quad = \quad \int_0^{\log 2} [(u'(x))^2 + (u(x) - 2)^2] \, \mathrm{d}x \tag{4}$$

subject to the conditions $u(\log 2) = 1$ and $2 \leq u(0) \leq 3$.

1. Can the solution be uniquely determined?
2. Let the additional condition $u'(0) = 0$ be given. Is a solution that you obtain from this condition unique?
3. Let the additional condition $u(0) = 2$ be given. Is a solution that you obtain from this condition unique?

Hint: A candidate function for the second derivative is $u''(x) = m(\exp(x) + \exp(-x))$ or $u''(x) = m(\exp(x) - \exp(-x))$ with a to-be-determined constant m.

Exercise No. 4 – Channel Reloaded (8 points)

Consider this variant of the channel example from the lecture

$$I_3(u) \quad = \quad \int_0^1 \sqrt{1 + (u'(x))^2} + z u(x) \, \mathrm{d}x. \tag{5}$$

for a multiplier factor $z \in \mathbb{R}$. Compute the Euler-Lagrange equation to obtain an optimal solution under the constraints $u(0) = u(1) = 0$.