The Fast Legendre Transform

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Outline

Application in Morphology

The Slope Transform

$$S(m) = stat_x(f(x) - mx)$$

The Slope Transform is in fact only a generalization of the Legendre Transform

- The Fourier Transform maps Convolution to Multiplication
- It has been shown that the Slope Transform is the morphological equivalent of the Fourier Transform
- It maps Dilation to Addition

The Legendre Transform can be used to speed up morphological operations like dilation.

Definition

The Legendre Transform ist defined by

$$f^*(m) = max_x(mx - f(x))$$

- $f^*(m)$ denotes the Legendre transform of m
- $max_x(...)$ maximizes the expression w.r.t. x
- m denotes the slope $f'(x_0)$ at some x_0
- $f(x)$ denotes a known (convex) function

Definition (2)

Reminder

A line t with slope m and y-interception n is given by the formula

$$t(x) = mx + n$$

Observation

The Legendre Transform measures the distances between the line $t(x) = mx$ and $f(x)$ and finally outputs the maximal distance.

Definition (3)

Is there a relation between m and the position x_0 where the maximum is obtained?

Computing

$$\frac{\partial}{\partial x_0}(mx_0 - f(x_0)) = 0$$

yields

$$m = f'(x_0)$$

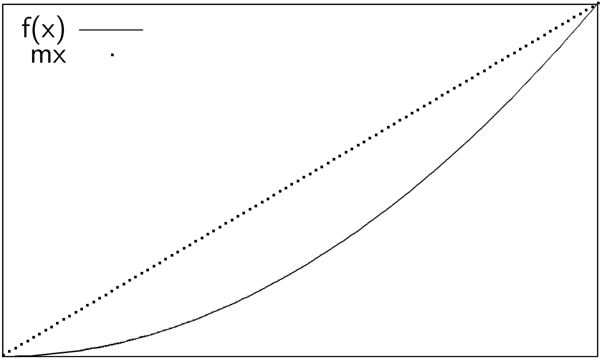
This enables us to state another version of the Legendre transform:

$$f^*(m) = mf'^{-1}(m) - f(f'^{-1}(m))$$

Interpretation of the Legendre Transform

Idea

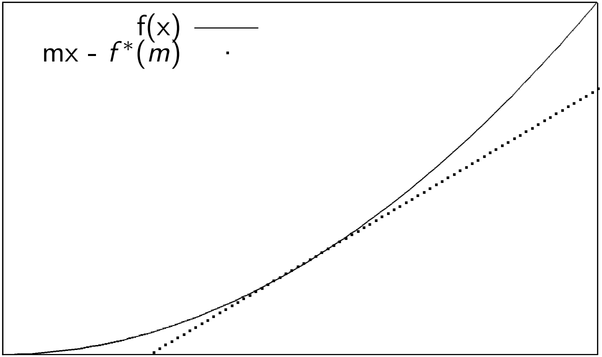
Consider:



Interpretation of the Legendre Transform (2)

Idea

Construct the line $mx - f^*(m)$:



Interpretation of the Legendre Transform (3)

Observation:

$f^*(m)$ computes the negative y-interception of the tangent of f with the slope m.

Proof:

A tangent at position x_0 has to fulfill the following constraint:

$$f(x_0) = f'(x_0)x_0 + n$$

This leads to

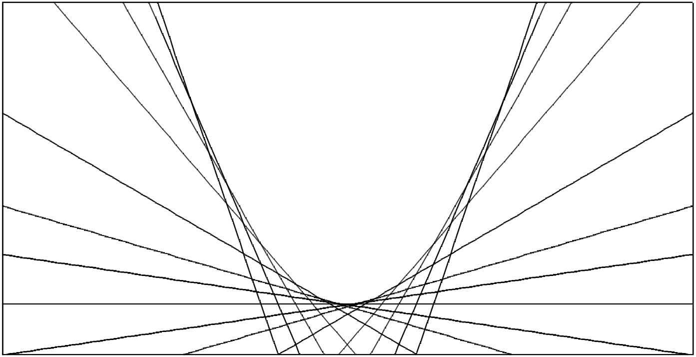
$$n = -f^*(f'(x_0))$$

By using the Legendre Transform we can parameterize the family of tangents of a graph:

$$F(x, y, m) = mx - y - f^*(m) = 0$$

Backtransformation

Given a family of tangents $F(x, y, m) = 0$ we want to find a function f which touches each member of this family. This function is also called the Envelope of F.
Example of a family of tangents:



Backtransformation (2)

Property of the envelope:

- Points on the envelope are points where infinitesimally adjacent members of F intersect
- This means x and y in $F(x, y, m)$ have to be constant in m at these points

This leads to the following 2 constraints:

$$F(x, y, m) = 0$$

$$\frac{\partial}{\partial m} F(x, y, m) = 0$$

Eliminating m from these 2 equations yields that the Legendre Transform is its own inverse:

$$f(x) = f^{**}(x)$$

Some concepts from convex analysis

Convex Set
A set of points is called convex if all points inside the set can be linked by a line without leaving the set.

Convex Hull
The set of points envelopping a convex set is called the convex hull.

The Epigraph of a function
The set of points lying on the graph and above it is called Epigraph.

Convex function
A function is convex if its Epigraph is a convex set.

Some observations

- It is only necessary to find x_0 where the maximum occurs
- A convex function has the property $f''(x) \geq 0$, i.e. its slope is only increasing
- A non-convex function can be made convex by applying a convex hull algorithm

So the Legendre Transform can be computed by solving the following problem:

$$H(m) = arg\ max_x(mx - f(x))$$

This function outputs the value x_0 where the expression is maximized.

Some observations (2)

Given n points $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$ we can compute the local slopes c_i by using the following formula:

$$\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

Since $f''(x) \geq 0$ the sequence c_i is only increasing.

Given m, computing $H(m)$ is now rather straightforward:

- If $c_{i-1} < m < c_i$ then $H(m) = \{x_i\}$
- If $c_i = m$ then $H(m) = \{x_i, x_{i+1}\}$

The Algorithm

Now we are ready to state the algorithm:

1. Input data: Choose $x_1, \ldots .x_n, f(x_1), \ldots, f(x_n)$ and $m_1, \ldots, m_m$
2. Convex step: Compute the convex hull of $(x_i, f(x_i))$ and rename the resulting sequence so that $P_1, \ldots, P_h$ are the vertices of the convex hull.
3. Merge Step: Compute c_i for $i = 1, \ldots, h$. Next, for each m_j, find the index i for which $c_i < m_j < c_{i+1}$

Complexity

This algorithm has a linear-time complexity.

Remarks

- One can prove that computing the dicrete Legendre Transform of a function converges towards the continuous one
- Note: $n = m$ is required to obtain good numerical accuracy
- The convex step of the algrithm is only necessary if the signal is non-convex
- Applying the algorithm twice to a non-convex signal yields the convex hull of the signal

Summary

- The Legendre Transform maps a function to its family of tangents
- It is its own inverse (only for convex functions)
- The presented algorithm transforms even non-convex signals
- The Legendre Transform can be used to speed up morphological operations like dilation

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