A Shock-Capturing Algorithm for the Differential Equations of Dilation and Erosion

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based on a paper by M. Breuß, J. Weickert

Outline:

- Introduction
- Reminder about Upwinding
- The FCT Scheme for 1-D Dilation
- The 2-D FCT Scheme, Examples
- Conclusions

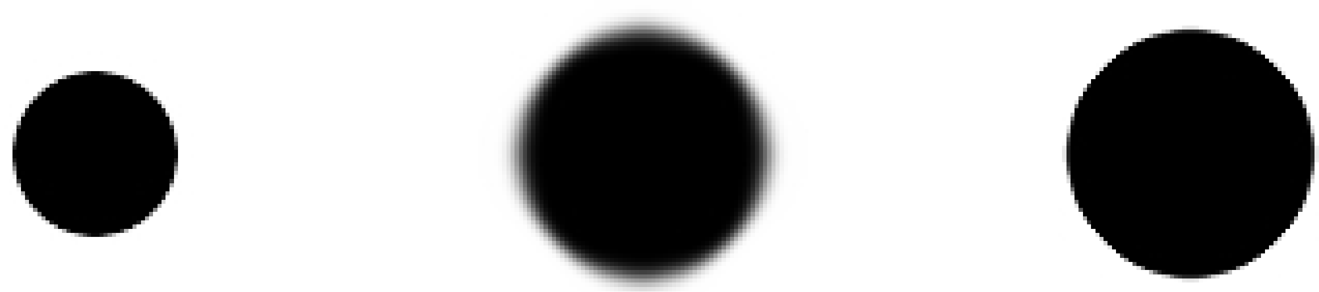
Introduction

- The two basic morphological processes of dilation and erosion (using a disc as structuring element) are described by the PDE

$$\partial_t u = \pm \|\nabla u\|_2$$

- Prominent numerical methods used to solve this PDE include the upwinding schemes proposed by Osher-Sethian and Rouy-Tourin.

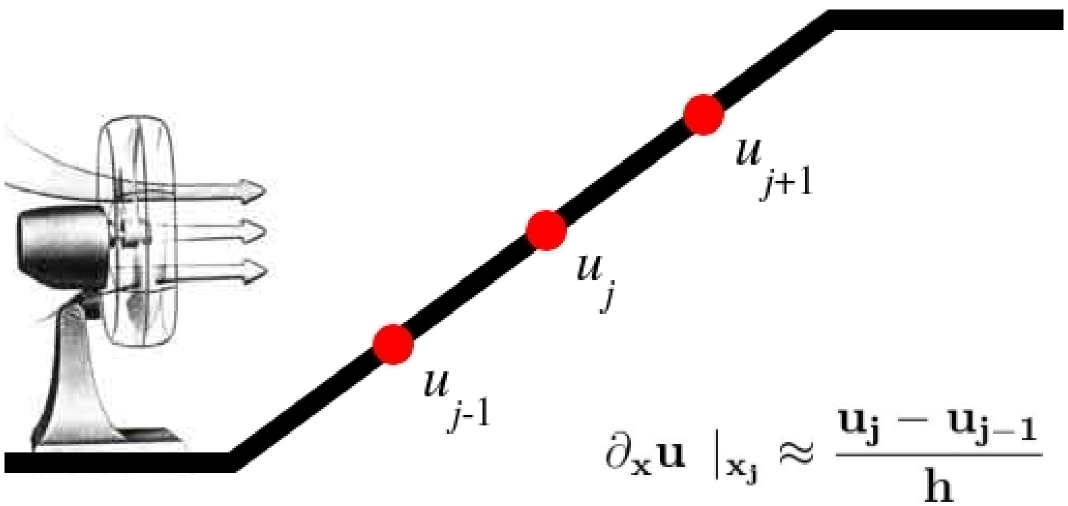
- Both suffer from undesirable blurring effects, also called numerical viscosity. This problem is overcome by the flux corrected transport (FCT) technique by Breuß-Weickert.



Left: Initial image. **Middle:** Erosion by Rouy-Tourin scheme.
Right: Erosion by FCT scheme (in both cases $\tau = 0.5$, 20 iterations).

Reminder about Upwinding

- The notion of "upwinding" comes from thinking of a sailboat which turns its sails into the direction of the wind. For numerical methods this means in analogy that derivatives are approximated by one-sided differences, in the direction from which information is coming.



Example of upwind scheme

- For a general one-dimensional hyperbolic first-order PDE $\partial_t u + \partial_x(f(u)) = 0$ the "direction of the wind" is to the right in case $f'(.) \geq 0$ and to the left in case $f'(.) < 0$.

- If the time step is chosen sufficiently small, the upwind scheme diminishes the number of extrema and satisfies a discrete maximum-minimum principle.

Basic Idea

- For the 1-D dilation equation $\partial_t u = \partial_x u$ the "blurry" Rouy-Tourin scheme uses the first-order approximation

$$\partial_x u \approx \frac{1}{h} \max(0,\ \delta U^n_{j+1/2},\ -\delta U^n_{j-1/2})$$

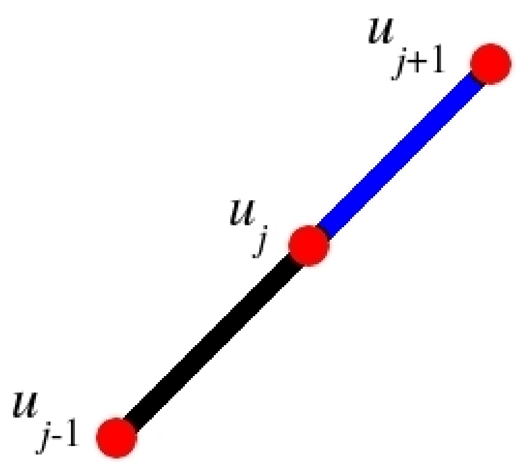
where $\delta U^n_{j+1/2} = U^n_{j+1} - U^n_j$ and $\delta U^n_{j-1/2} = U^n_j - U^n_{j-1}$.

- The basic idea behind the FCT scheme is to rewrite these one-sided discrete differences (which are a first order approximation of $\partial_x u$) as the sum of a second order approximation of $\partial_x u$ and a so called viscosity term. This term is responsible for the blurring artifacts in the results.

- Therefore what can be done is to:

1. calculate the value of U^{n+1}_j by the Rouy-Tourin scheme in an initial predictor step.
2. subtract the viscosity term from U^{n+1}_j in a corrector step.

Different Cases: Case I

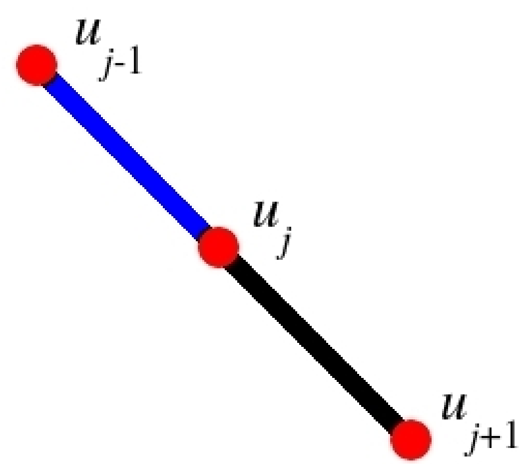


- If $\delta U^n_{j-1/2} \geq 0$ and $\delta U^n_{j+1/2} > 0$ the upwind scheme reads:

$$\begin{aligned}\frac{U_j^{n+1} - U_j^n}{\tau} &= \frac{U^n_{j+1} - U^n_j}{h} \\ &= \underbrace{\frac{U^n_{j+1} - U^n_{j-1}}{2h}}_{(a)} + \underbrace{\frac{U^n_{j+1} - U^n_j}{2h} - \frac{U^n_j - U^n_{j-1}}{2h}}_{(b)}\end{aligned}$$

where (a) is a second order approximation of $|\partial_x u|$ and (b) is the viscosity term.

Different Cases: Case II

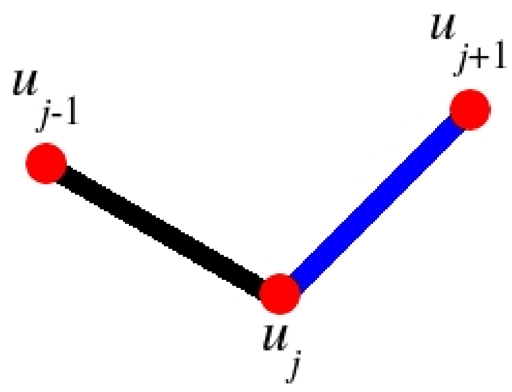


- If $\delta U^n_{j-1/2} < 0$ and $\delta U^n_{j+1/2} \leq 0$ the upwind scheme reads:

$$\frac{U_j^{n+1} - U_j^n}{\tau} = -\frac{U_j^n - U_{j-1}^n}{h}$$
$$= \underbrace{\frac{U_{j-1}^n - U_{j+1}^n}{2h}}_{(a)} + \underbrace{\frac{U_{j+1}^n - U_j^n}{2h} - \frac{U_j^n - U_{j-1}^n}{2h}}_{(b)}$$

where (a) is a different second order approximation of $|\partial_x u|$ but the viscosity term (b) is the same as in "Case I".

Different Cases: Case III - Local Minima

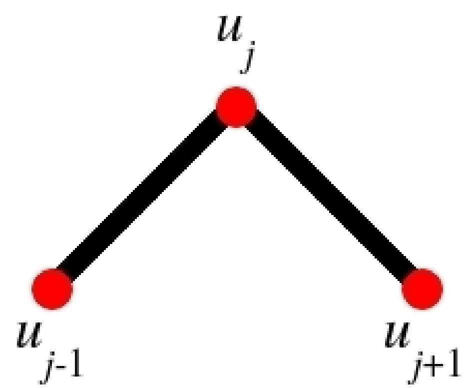


- If $\delta U^n_{j-1/2} < 0$ and $\delta U^n_{j+1/2} \geq 0$ the upwind scheme reads:

$$\frac{U_j^{n+1} - U_j^n}{\tau} = \max\left(\frac{U_{j+1}^n - U_j^n}{h}, -\frac{U_j^n - U_{j-1}^n}{h}\right)$$

where we have the same viscosity form as in “Case I” and “Case II” respectively.

Different Cases: Case IV - Local Maxima



- If $\delta U^n_{j-1/2} \geq 0$ and $\delta U^n_{j+1/2} \leq 0$ the upwind scheme reads:

$$\frac{U_j^{n+1} - U_j^n}{\tau} = 0$$

according to the principle that local maxima are maintained.

Summary of Cases I to IV

- Cases I to IV can be summarized as:

$$U_j^{n+1} = \begin{cases} U_j^n & \text{for local maxima} \\ U_j^n + \frac{\lambda}{2}|\Delta U_j^n| + \frac{\lambda}{2}\delta U_{j+1/2}^n - \frac{\lambda}{2}\delta U_{j-1/2}^n & \text{else} \end{cases}$$

where $\lambda = \frac{\tau}{h}$ and $\Delta U_j^n = U_{j+1}^n - U_{j-1}^n$.

- This scheme is identical to the Rouy-Tourin method, but we have gained that we can now identify the numerical viscosity arising by the first order approximation of the spatial derivative.

- Let us define:
  - $U_j^{n+1/2}$ as the data obtained by the upwind scheme starting from U_j^n
  - U_j^{n+1} as the data obtained after subtracting the viscosity term from $U_j^{n+1/2}$

- Simply subtracting the viscosity term from $U_j^{n+1/2}$ would lead to unstable evolutions, therefore we have to introduce the function g defined as:

$$g_{j+1/2} := \operatorname{minmod}\left(\delta U_{j-1/2}^{n+1/2},\ \frac{\lambda}{2}\,\delta U_{j+1/2}^{n+1/2},\ \delta U_{j+3/2}^{n+1/2}\right)$$

$$\operatorname{minmod}(a,b,c) := \operatorname{sign}(b)\max\Big(0,\ \min(\operatorname{sign}(b)\cdot a,\ |b|,\ \operatorname{sign}(b)\cdot c)\Big)$$

  The middle argument of g corresponds indeed to the viscosity term, whereas the left and right arguments are responsible for stabilization.

- Finally U_j^{n+1} is given by

$$U_j^{n+1} = U_j^{n+1/2} - g_{j+1/2} + g_{j-1/2}$$

- **Stability Result:** if $\tau \le h$ is chosen, the investigated scheme satisfies locally and globally a discrete maximum-minimum principle.

- The extension of the one-dimensional analysis to the two-dimensional dilation/erosion PDE

$$\partial_t = \pm \|\nabla u\|_2 = \sqrt{|\partial_x u|^2 + |\partial_y u|^2}$$

is straightforward.

- Again, the idea is to separate the viscosity term from a second-order discretization of $|\partial_x u|$ and $|\partial_y u|$.

- **Stability Result:** if $\tau \leq \frac{h}{\sqrt{2}}$ is chosen, also the 2-D FCT scheme satisfies locally and globally a discrete maximum-minimum principle.

Examples

Left: Initial image. **Middle:** Erosion by FCT scheme.
Right: Dilation by FCT scheme (in both cases $\tau = 0.5, 3$ iterations).

Examples

Left: Initial image. **Middle:** Beucher gradient by Rouy-Tourin scheme.
Right: Beucher gradient by FCT scheme (in both cases $\tau = 0.5, 2$ iterations).

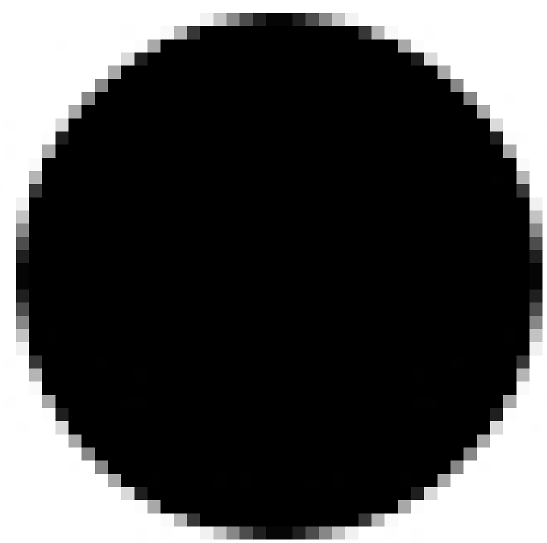
- It is possible to see that the Beucher gradient (difference between dilated and eroded image) calculated by the FCT scheme appears less blurred than the Beucher gradient calculated by the Rouy-Tourin scheme.

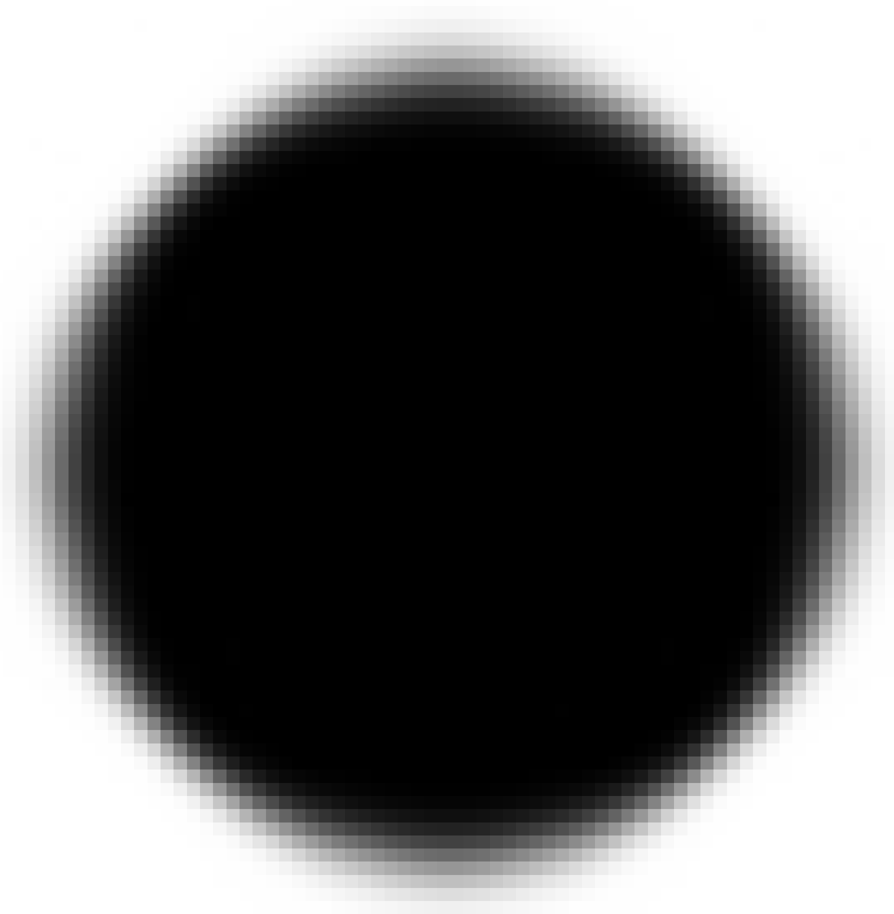
Conclusions:

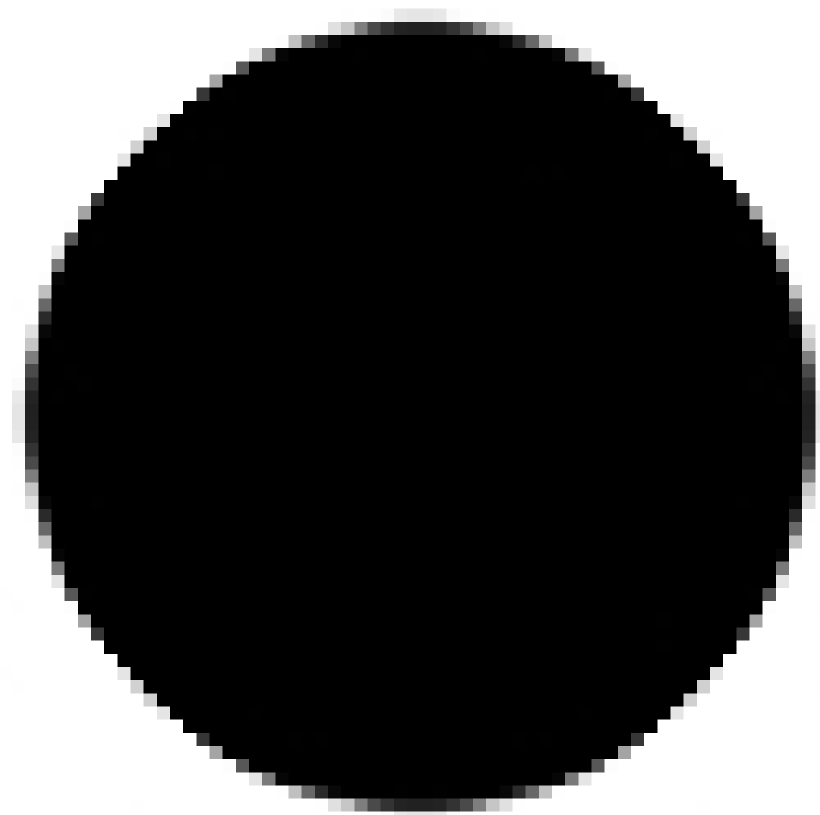
- The FCT scheme by Breuß-Weickert overcomes blurring problems typical of classical upwind schemes.
- The basic idea for deriving the FCT scheme is to rewrite the one-sided discrete differences (which approximate $\partial_x u$) as the sum of a second order approximation of $\partial_x u$ and a so called viscosity term, which will later be subtracted from the solution.

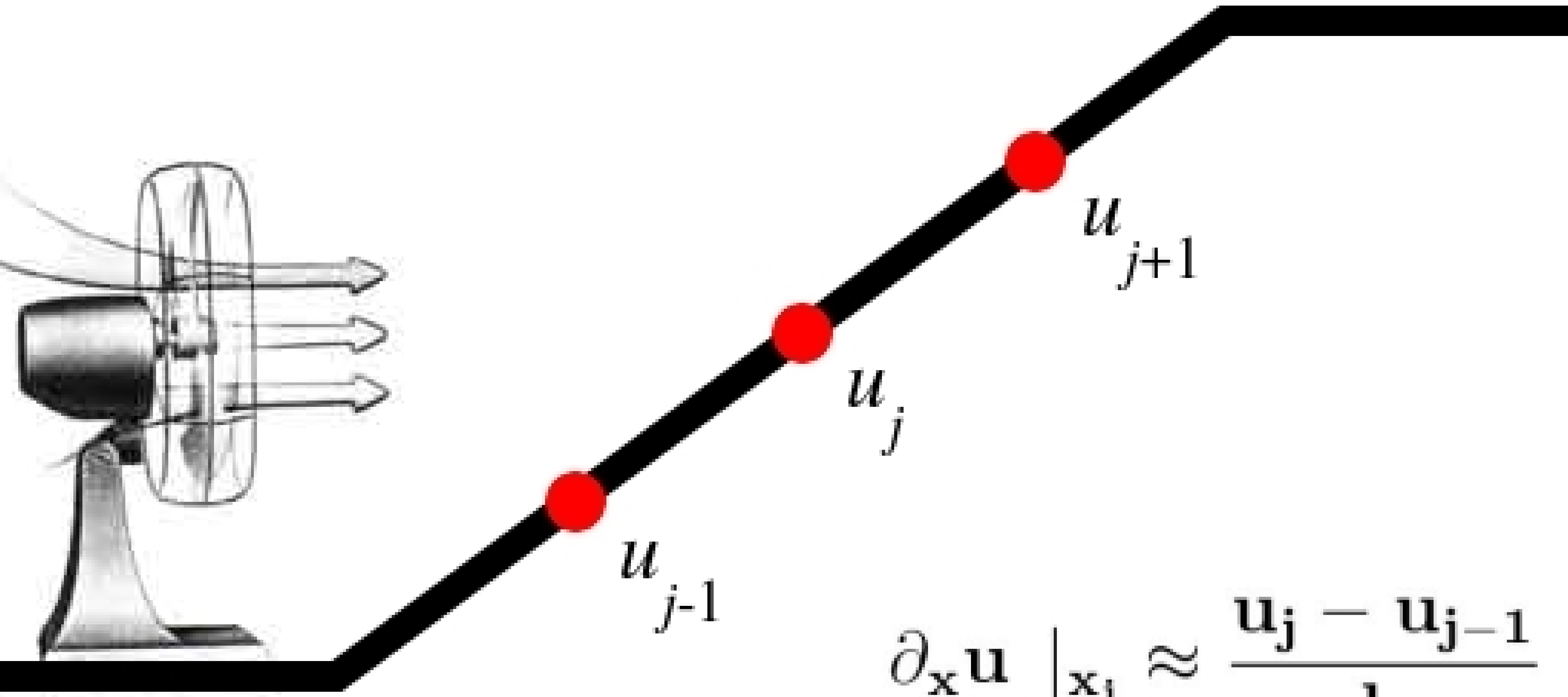
Bibliography

1. M. Breuß, J. Weickert:
   A shock-capturing algorithm for the differential equations of dilation and erosion.
   Journal of Mathematical Imaging and Vision, Vol. 25, 187-201, 2006.

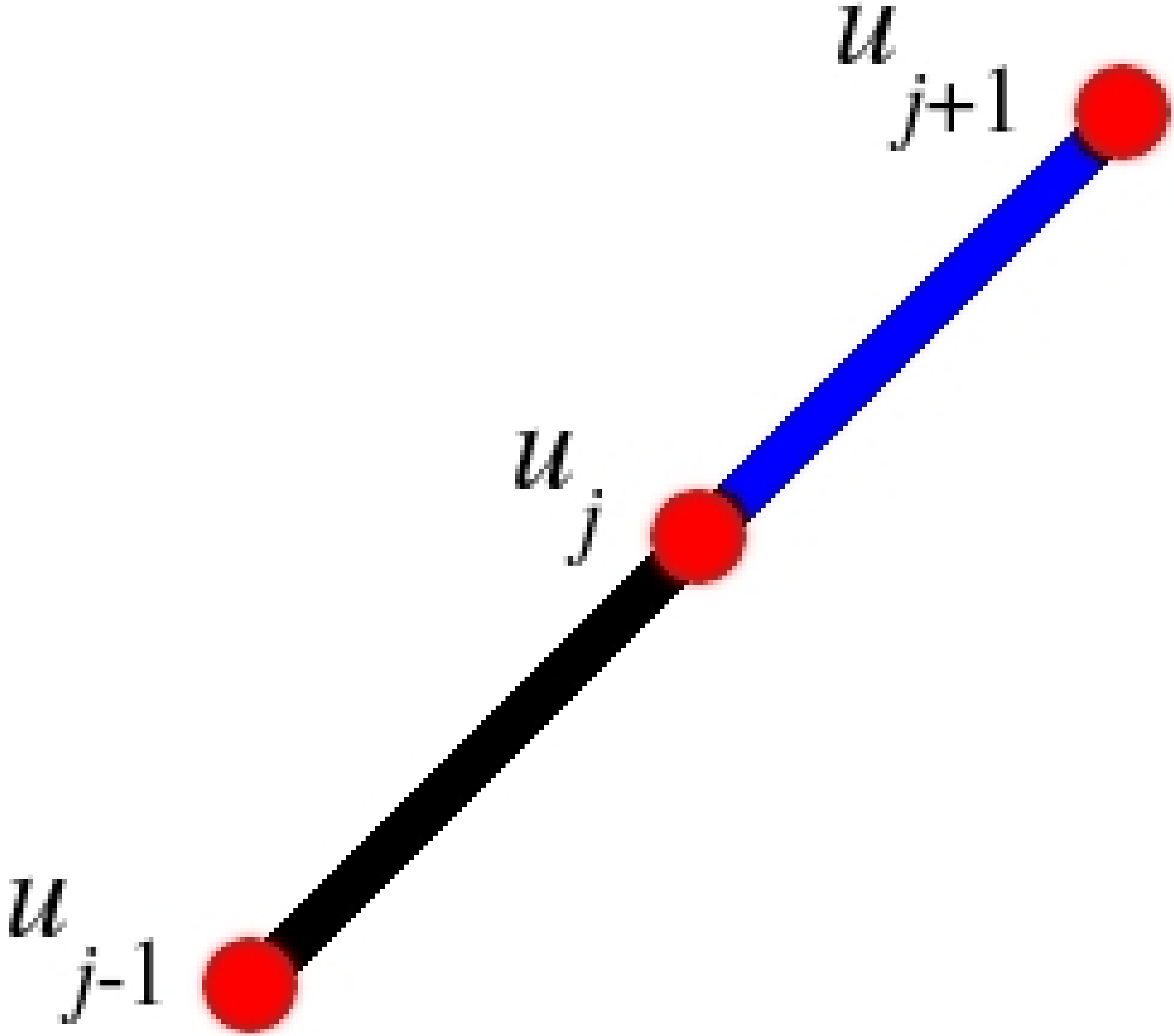


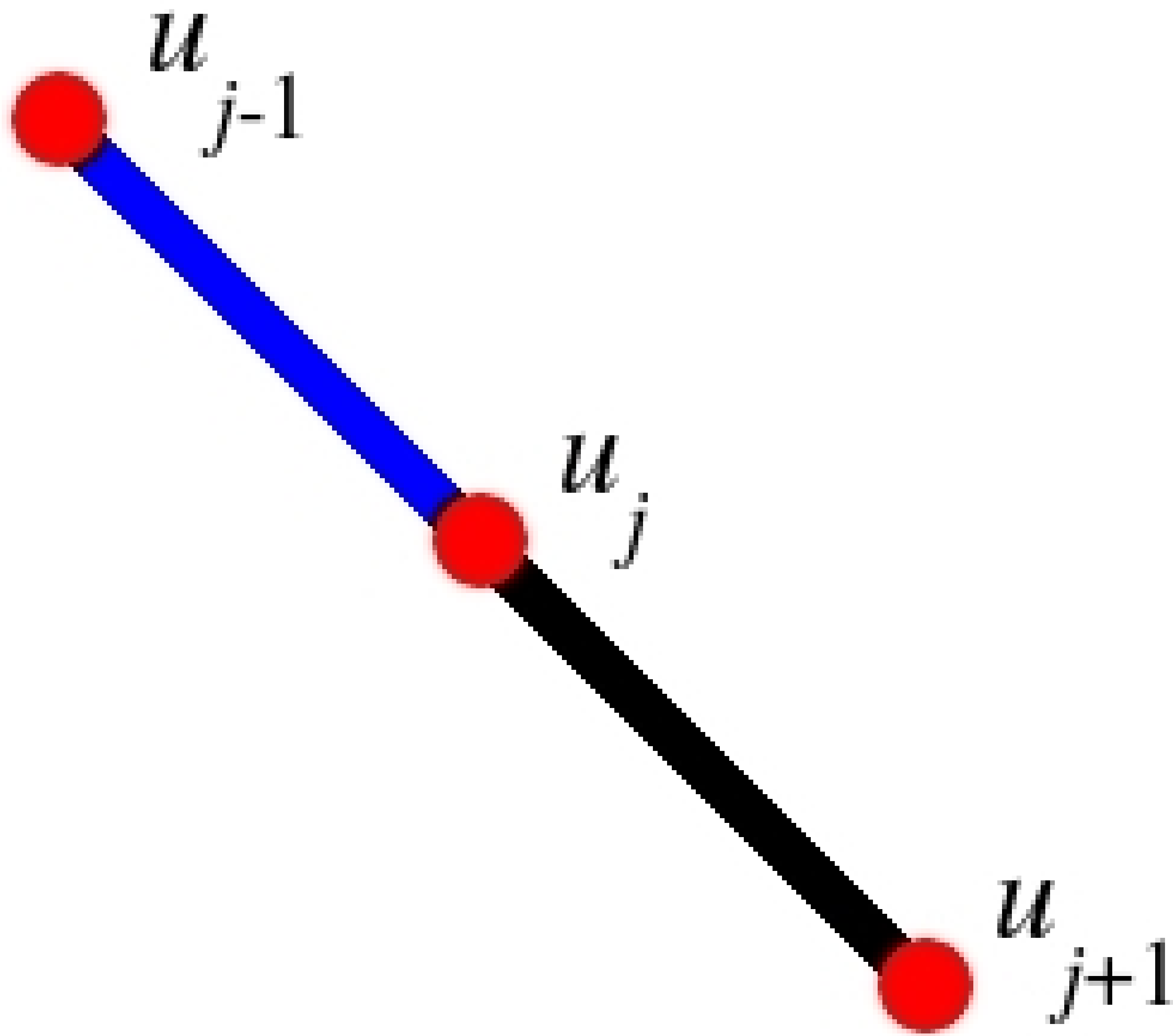


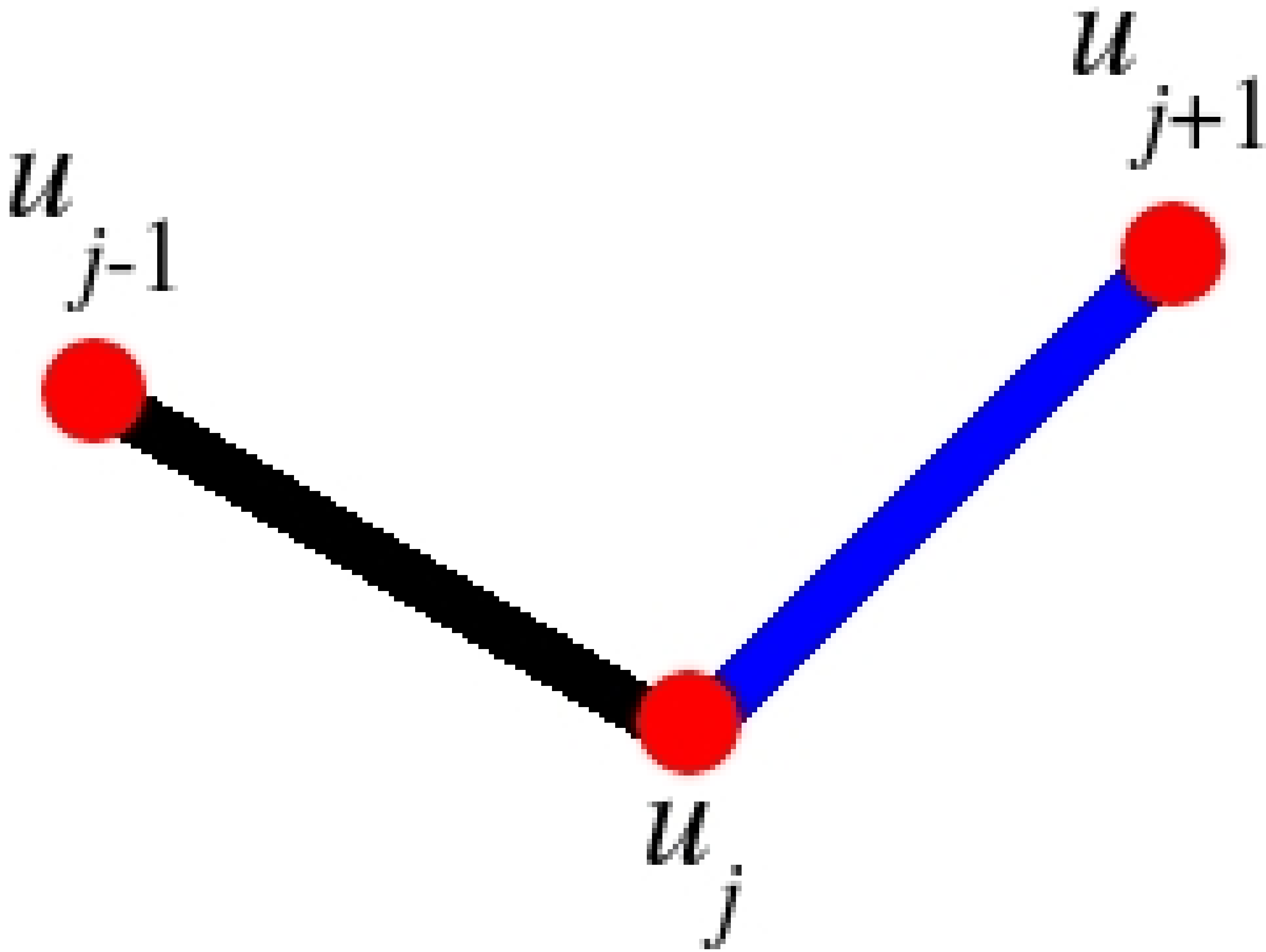


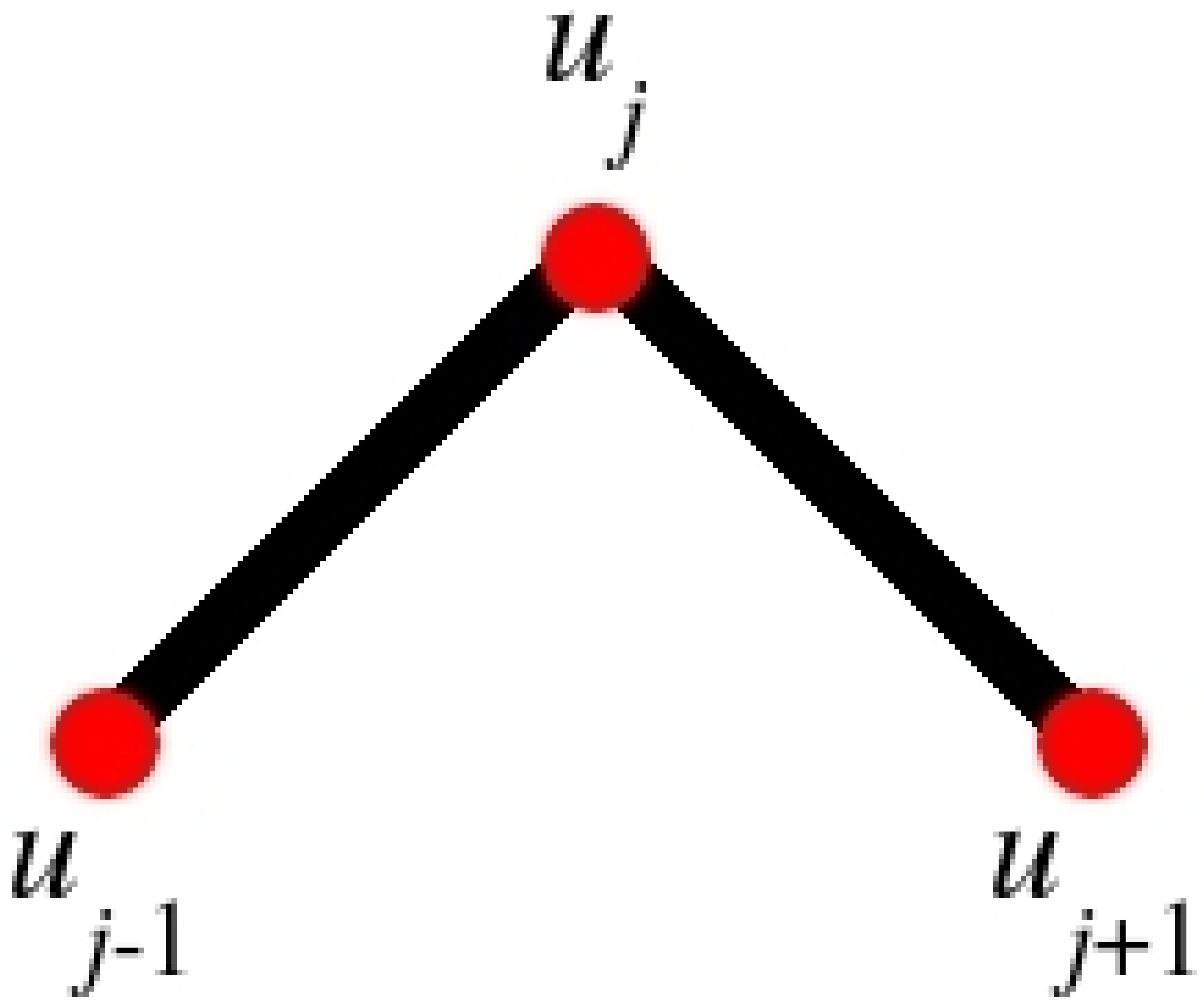


$$\partial_{\mathbf{x}} \mathbf{u} \mid_{\mathbf{x}_{\mathbf{j}}} \approx \frac{\mathbf{u}_{\mathbf{j}} - \mathbf{u}_{\mathbf{j-1}}}{\mathbf{h}}$$


u_{j+1}
u_j
u_{j-1}


u_{j-1}
u_j
u_{j+1}


u_{j-1}
u_{j+1}
u_j



u_j
u_{j-1}
u_{j+1}