Mathematical Foundations of Computer Vision

Example Solution – Assignment 8

Solution of Exercise No. 1

Let the essential matrix

$$E = \begin{pmatrix} 0.76 & 4.32 & -2.4 \\ -4.32 & 1.76 & 1.8 \\ 0 & 0 & 0 \end{pmatrix} \tag{1}$$

be given. A singular value decomposition for E is

$$E = \begin{pmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0.64 & 0.48 & -0.6 \\ -0.6 & 0.8 & 0 \\ 0.48 & 0.36 & 0.8 \end{pmatrix} \tag{2}$$

Compute the epipoles in both images. Give them in homogeneous coordinates.

We have $E = U\Sigma V^\top$. We also have the relations

$$(i) \quad E\vec{e}_1 = \vec{0} \quad \text{and} \quad (ii) \quad E^\top \vec{e}_1 = \vec{0}$$

for the two epipoles $\vec{e}_1$, $\vec{e}_2$.

(i) The relation $U\Sigma V^\top e_1 = \vec{0}$ is satisfied for for e_1 being identical to the last row of $V^\top$, because with $V = [v_1, v_2, v_3]$ we always obtain

$$U\Sigma[v_1, v_2, v_3]^\top v_3 = U\Sigma \begin{pmatrix} \langle v_1, v_3\rangle \\ \langle v_2, v_3\rangle \\ \langle v_3, v_3\rangle \end{pmatrix} = U\Sigma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = U \begin{pmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = U\vec{0} = \vec{0}$$

Therefore

$$\vec{e}_1 = \begin{pmatrix} 0.48 \\ 0.36 \\ 0.8 \end{pmatrix} \sim \begin{pmatrix} 0.6 \\ 0.45 \\ 1 \end{pmatrix}$$

(ii) The relation

$$\left(U\Sigma V^\top\right)^\top e_2 = \vec{0} \quad \Leftrightarrow \quad V\Sigma U^\top e_2 = \vec{0}$$

holds, if analogously to the procedure above $\vec{e}_2$ is given by the last row of $U^\top$, i.e. $\vec{e}_2 = (0, 0, 1)^\top$.

Solution of Exercise No. 2

Consider homogeneous transformations H (expressed as 3×3-matrix) in the (projective) image plane. Determine H such that the points with homogeneous coordinates

$$\vec{a} = (0,0,1)^\top, \quad \vec{b} = (1,0,1)^\top, \quad \vec{c} = (1,1,1)^\top, \quad \vec{d} = (0,1,1)^\top \tag{3}$$

are mapped to the points with homogeneous coordinates

$$\vec{a}' = (6,5,1)^\top, \quad \vec{b}' = (4,3,1)^\top, \quad \vec{c}' = (6,4.5,1)^\top, \quad \vec{d}' = (10,8,1)^\top \tag{4}$$

respectively.

In order to determine H, we make use of the constraints

$$H\vec{a} = \alpha\vec{a}', \quad H\vec{b} = \beta\vec{b}', \quad H\vec{c} = \gamma\vec{c}', \quad H\vec{d} = \delta\vec{d}'$$

wher we have introduced as appropriate for the homogeneous setting the unknown scaling parameters $\alpha, \beta, \gamma, \delta$.

This results in the following system of equations:

$$\begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 6\alpha & 4\beta & 6\gamma & 10\delta \\ 5\alpha & 3\beta & 4.5\gamma & 8\delta \\ \alpha & \beta & \gamma & \delta \end{pmatrix}$$

We now head for an adequate formulation enabling the numerical solution of the problem by standard numerical software.

Stacking the unknowns – i.e. the nine entries of H columnwise, plus the four scaling variables – into a vector v, we obtain $Gv = \vec{0}$ where

$$G = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 4.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We have 13 unknowns and 12 equations, i.e. H is only determined up to a scale.

Solving this system of equations and expressing all entries of H in terms of α gives

$$H = \alpha \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 1 & 0 & 1 \end{pmatrix}$$