Mathematical Foundations of Computer Vision

Example Solution – Assignment 7

Solution of Exercise No. 1

Assume that the following essential matrix is given:

$$E = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & 2 & 0 \end{pmatrix} \tag{1}$$

Compute the correct pose (R, T) from E.

First, we compute the SVD of

$$E = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & 2 & 0 \end{pmatrix}.$$

- computation of $B = E^T E$:

$$E^T E = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & 2 \\ 0 & \sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

- eigenvalue of $E^T E$: $\lambda_{1,2} = 4 > 0$ and $\lambda_3 = 0$, because

$$\begin{aligned} \det(B - \lambda I) &= \begin{vmatrix} 2-\lambda & 0 & 2 \\ 0 & 4-\lambda & 0 \\ 2 & 0 & 2-\lambda \end{vmatrix} \\ &= (2-\lambda)^2(4-\lambda) - 4(4-\lambda) \\ &= (4 - 4\lambda + \lambda^2)(4-\lambda) - 16 + 4\lambda \\ &= -\lambda^3 + 8\lambda^2 - 16\lambda \\ &= -\lambda(\lambda^2 - 8\lambda + 16) \\ &= -\lambda(\lambda - 4)^2 \end{aligned}$$

- matrix V contains the normalized eigenvectors of $E^T E$:

$$\begin{aligned} \text{Eig}(B, 4) &= \ker \begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & -2 \end{pmatrix} \\ &=< \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} > \\ &=< v_1, v_2 > \end{aligned}$$

$$\begin{aligned} \text{Eig}(B, 0) &= \ker \begin{pmatrix} 2 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \\ &= v_3 \end{aligned}$$

Now, we normalise the vectors v_1, v_2 and v_3:

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$
$$v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$
$$v_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

This leads to the matrix

$$\begin{aligned} V &= (v_1|v_2|v_3) \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \end{aligned}$$

- diagonal matrix Σ:

$$\Sigma = diag(\sqrt{\lambda_1}, \sqrt{\lambda_2}, \sqrt{\lambda_3}) = diag(2, 2, 0) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- matrix U:

$$\begin{aligned} u_1 &:= \frac{1}{\sqrt{\lambda_1}} E v_1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & 2 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ u_2 &:= \frac{1}{\sqrt{\lambda_2}} E v_2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ \Rightarrow u_3 &:= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ \Rightarrow U &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \end{aligned}$$

We now compute the four possible poses (R_i, T_j), $i, j = 1, 2$. Plugging in the corresponding matrices we obtain:

$$
\begin{aligned}
R_1 = U R_z^\top\left(+\frac{\pi}{2}\right) V^\top &= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & -1 \end{pmatrix} \\
R_2 = U R_z^\top\left(-\frac{\pi}{2}\right) V^\top &= \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 0 & 1 \\ 0 & -\sqrt{2} & 0 \\ 1 & 0 & -1 \end{pmatrix} \\
\hat{T}_1 = U R_z\left(+\frac{\pi}{2}\right) \Sigma U^\top &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \quad \text{i.e. } T_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \\
\hat{T}_2 = U R_z\left(-\frac{\pi}{2}\right) \Sigma U^\top &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}, \quad \text{i.e. } T_2 = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}
$$

What is the correct pose?

We generate a correspondence pair as follows:

- We test for the optical axis, setting $\vec{x}_1 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
- For the epipolar constraint we obtain

$$
\vec{x}_2^\top E \vec{x}_1 = \vec{x}_2^\top \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \quad \text{satisfied e.g. by } \vec{x}_2 := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

Therefore a correspondence pair (which has the character of a test pair) is given by intersection of the two optical axes.

We now invoke the positive depth constraint:

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$$
\begin{aligned}
(R_1, T_1) \quad : \quad \lambda_1 \begin{pmatrix} -1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} &= \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\
\Leftrightarrow \begin{pmatrix} \frac{\lambda_1}{\sqrt{2}} + 2 \\ 0 \\ \frac{-\lambda_1}{\sqrt{2}} \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned}
$$

  which implies that λ_1 is negative, so that we discard the pose (R_1, T_1).

- An analogous computation gives

$$
(R_1, T_2) \quad : \quad \begin{pmatrix} \frac{\lambda_1}{\sqrt{2}} - 2 \\ 0 \\ \frac{-\lambda_1}{\sqrt{2}} \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

  which implies that λ_2 is negative, so that we discard the pose (R_1, T_2).

- An analogous computation gives

$$
(R_2, T_1) \quad : \quad \begin{pmatrix} \frac{\lambda_1}{\sqrt{2}} + 2 \\ 0 \\ \frac{\lambda_1}{\sqrt{2}} \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

  which implies that both λ_1 and λ_2 are negative, so that we discard the pose (R_2, T_1).

- An analogous computation gives

$$(R_2, T_2) \quad : \quad \begin{pmatrix} \frac{\lambda_1}{\sqrt{2}} - 2 \\ 0 \\ \frac{\lambda_1}{\sqrt{2}} \end{pmatrix} = \lambda_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

which shows that both λ_1 and λ_2 are positive, so that (R_2, T_2) is the correct pose.