Mathematical Foundations of Computer Vision

Example Solutions – Assignment 6

Solution of Exercise No. 1

(a) Validate for $\vec{w}$ being of unit length, that (i) $\hat{w}^2 = \vec{w}\vec{w}^\top - I$ *and* (ii) $\hat{w}^3 = -\hat{w}$.

(i) For $w = (w_1, w_2, w_3)$ with $||w|| = \sqrt{w_1^2 + w_2^2 + w_3^2} = 1$, the matrix of the Cross-Product ist given by

$$\hat{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

Now, we compute $\hat{w}^2$:

$$\begin{aligned} \hat{w}^2 &= \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -w_2^2 - w_3^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & -w_1^2 - w_3^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & -w_2^2 - w_1^2 \end{pmatrix} \end{aligned}$$

We know, that $\sqrt{w_1^2 + w_2^2 + w_3^2} = 1 \Longleftrightarrow w_1^2 + w_2^2 + w_3^2 = 1$ and we can use this, to get:

$$\begin{aligned} \hat{w}^2 &= \begin{pmatrix} -w_2^2 - w_3^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & -w_1^2 - w_3^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & -w_2^2 - w_1^2 \end{pmatrix} \\ &= \begin{pmatrix} w_1^2 - 1 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 - 1 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 - 1 \end{pmatrix} \\ &= \begin{pmatrix} w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 \end{pmatrix} - I \\ &= \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix} - I \\ &= w w^\top - I \end{aligned}$$

(ii) Now, we compute $\hat{w}^3$:

$$
\begin{aligned}
\hat{w}^3 &= \hat{w}^2 \cdot \hat{w} \\
&= (ww^\top - I) \cdot \hat{w} \\
&= \left(\begin{pmatrix} w_1^2 & w_1w_2 & w_1w_3 \\ w_1w_2 & w_2^2 & w_2w_3 \\ w_1w_3 & w_2w_3 & w_3^2 \end{pmatrix} - I \right) \cdot \hat{w} \\
&= \begin{pmatrix} w_1^2 & w_1w_2 & w_1w_3 \\ w_1w_2 & w_2^2 & w_2w_3 \\ w_1w_3 & w_2w_3 & w_3^2 \end{pmatrix} \cdot \hat{w} - I \cdot \hat{w} \\
&= \begin{pmatrix} w_1^2 & w_1w_2 & w_1w_3 \\ w_1w_2 & w_2^2 & w_2w_3 \\ w_1w_3 & w_2w_3 & w_3^2 \end{pmatrix} \cdot \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix} - \hat{w} \\
&= \begin{pmatrix} w_1w_2w_3 - w_1w_2w_3 & w_1^2w_3 - w_1^2w_3 & w_1^2w_2 - w_1^2w_2 \\ w_3w_2^2 - w_3w_2^2 & w_1w_2w_3 - w_1w_2w_3 & w_1w_2^2 - w_1w_2^2 \\ w_2w_3^2 - w_2w_3^2 & w_1w_3^2 - w_1w_3^2 & w_1w_2w_3 - w_1w_2w_3 \end{pmatrix} - \hat{w} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \hat{w} \\
&= -\hat{w}
\end{aligned}
$$

(b) Show that it holds:

$$
e^{\hat{w}t} \;=\; I + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} \mp \cdots \right) \hat{w} + \left(\frac{t^2}{2!} - \frac{t^4}{4!} + \frac{t^6}{6!} \mp \cdots \right) \hat{w}^2
$$

It holds:

$$
\begin{aligned}
\exp(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \ldots \\
\sin(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots \\
\cos(x) &= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots
\end{aligned}
$$

We can use this to compute $e^{\hat{w}t}$:

$$
\begin{aligned}
e^{\hat{w}t} &= \sum_{n=0}^{\infty} \frac{(\hat{w}t)^n}{n!} \\
&= I + \frac{\hat{w}t}{1!} + \frac{\hat{w}^2t^2}{2!} + \frac{\hat{w}^3t^3}{3!} + \frac{\hat{w}^4t^4}{4!} + \frac{\hat{w}^5t^5}{5!} + \frac{\hat{w}^6t^6}{6!} + \ldots \\
&= I + \left(\frac{t}{1!}\hat{w} + \frac{t^3}{3!}\hat{w}^3 + \frac{t^5}{5!}\hat{w}^5 + \ldots \right) + \left(\frac{t^2}{2!}\hat{w}^2 + \frac{t^4}{4!}\hat{w}^4 + \frac{t^6}{6!}\hat{w}^6 + \ldots \right) \\
&\overset{a)ii)}{=} I + \left(\frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} \mp \ldots \right) \hat{w} + \left(\frac{t^2}{2!} - \frac{t^4}{4!} + \frac{t^6}{6!} \mp \ldots \right) \hat{w}^2
\end{aligned}
$$

This is possible, because for $n \geqslant 0$ it holds:

$$\begin{aligned}
\hat{w}^3 &= -\hat{w} \\
\hat{w}^4 &= \hat{w}^3 \cdot \hat{w} = -\hat{w} \cdot \hat{w} = -\hat{w}^2 \\
\hat{w}^5 &= \hat{w}^3 \cdot \hat{w}^2 = -\hat{w} \cdot \hat{w}^2 = -\hat{w}^3 = -(-\hat{w}) = \hat{w} \\
\hat{w}^6 &= \hat{w}^3 \cdot \hat{w}^3 = (-\hat{w}) \cdot (-\hat{w}) = \hat{w}^2 \\
\hat{w}^7 &= \hat{w}^4 \cdot \hat{w}^3 = -\hat{w}^2 \cdot (-\hat{w}) = \hat{w}^3 = -\hat{w} \\
&\vdots \\
\hat{w}^{4n} &= -\hat{w}^2 \\
\hat{w}^{4n+1} &= \hat{w} \\
\hat{w}^{4n+2} &= \hat{w}^2 \\
\hat{w}^{4n+3} &= -\hat{w}
\end{aligned}$$

(c) Prove that the formula of Rodrigues holds true.

We have:

$$\begin{aligned}
e^{\hat{w}t} &= I + \left(\frac{t}{1!} - \frac{t^3}{3!} + \frac{t^5}{5!} \mp \dots \right) \hat{w} + \left(\frac{t^2}{2!} - \frac{t^4}{4!} + \frac{t^6}{6!} \mp \dots \right) \hat{w}^2 \\
&= I + \sin(t)\hat{w} + \left(1 - \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \pm \dots \right) \right) \hat{w}^2 \\
&= I + \sin(t)\hat{w} + (1 - \cos(t))\hat{w}^2
\end{aligned}$$

With $\hat{w} \mapsto \dfrac{\hat{w}}{t}$ it follows:

$$\begin{aligned}
e^{\hat{w}t} &= I + \sin(t)\hat{w} + (1 - \cos(t))\hat{w}^2 \\
\iff e^{\frac{\hat{w}}{t}t} &= I + \sin(t)\frac{\hat{w}}{t} + (1 - \cos(t)) \left(\frac{\hat{w}}{t} \right)^2 \\
\iff e^{\hat{w}} &= I + \sin(t)\frac{\hat{w}}{t} + (1 - \cos(t))\frac{\hat{w}^2}{t^2}
\end{aligned}$$

For $t := ||\vec{w}||$, it follows:

$$\begin{aligned}
e^{\hat{w}} &= I + \sin(t)\frac{\hat{w}}{t} + (1 - \cos(t))\frac{\hat{w}^2}{t^2} \\
\iff e^{\hat{w}} &= I + \sin(||\vec{w}||)\frac{\hat{w}}{||\vec{w}||} + (1 - \cos(||\vec{w}||))\frac{\hat{w}^2}{||\vec{w}||^2} \\
\iff e^{\hat{w}} &= I + \frac{\hat{w}}{||\vec{w}||}\sin(||\vec{w}||) + \frac{\hat{w}^2}{||\vec{w}||^2}(1 - \cos(||\vec{w}||))
\end{aligned}$$

Solution of Exercise No. 2

Theorem 1 *Given any $R \in SO(3)$, there exists a (in general, not unique) vector $v \in \mathbb{R}^3$ such that $R = e^{\hat{v}}$. We denote the inverse of the exponential map as $\hat{v} = \log(R)$.*

Prove the theorem.

We have to show the existence of such a vector v.

The proof is by construction: If the rotation matrix $R \neq I$ is given as

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

the corresponding ω is given by

$$\|\omega\| = \cos^{-1}\left(\frac{\text{trace}(R) - 1}{2}\right), \qquad \frac{\omega}{\|\omega\|} = \frac{1}{2\sin(\|\omega\|)} \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix}$$

If $R = I$, then $\|\omega\| = 0$, and $\frac{\omega}{\|\omega\|}$ is not determined, and therefore can be chosen arbitrarily.

Solution of Exercise No. 3

Prove or disprove for $\vec{w} \neq \vec{0}$

$$e^{\hat{\xi}} = \begin{pmatrix} e^{\hat{w}} & \frac{(I - e^{\hat{w}})\hat{w}v + \vec{w}\vec{w}^\top v}{\|w\|} \\ \vec{0}^\top & 1 \end{pmatrix} \tag{1}$$

where $v(t) = \dot{T}(t) - \hat{w}T(t)$.

Let us note, that for the twist coordinates we deal with in this assignment holds

$$\hat{\xi} = \begin{pmatrix} \hat{\omega} & v \\ \vec{0}^\top & 0 \end{pmatrix}$$

where the vector v is given as above.

By definition we have

$$e^{\hat{\xi}t} = I + \hat{\xi}t + \frac{\left(\hat{\xi}t\right)^2}{2!} + \frac{\left(\hat{\xi}t\right)^3}{3!} + \dots$$

By computing the first terms in this series it immediately follows that all entries in the matrix

$$\begin{pmatrix} e^{\hat{w}} & \frac{(I - e^{\hat{w}})\hat{w}v + \vec{w}\vec{w}^\top v}{\|w\|} \\ \vec{0}^\top & 1 \end{pmatrix}$$

are correct for a unit twist $\|\xi\| = 1$, but except for the entry

$$\frac{(I - e^{\hat{w}})\hat{w}v + \vec{w}\vec{w}^\top v}{\|w\|}$$

on which we focus now.

Computing the first terms in the exponential series above shows that we have at the corresponding place the series

$$\left(\frac{t}{1!} + \hat{\omega}\frac{t^2}{2!} + \hat{\omega}^2\frac{t^3}{3!} + \dots\right) v$$

which shall be made identical to the remaining entry on the right hand side of (1).

By adding zero we obtain

$$\left(\frac{t}{1!} + \hat{\omega}\frac{t^2}{2!} + \hat{\omega}^2\frac{t^3}{3!} + \dots\right) v \;=\; \left(\frac{t}{1!} + \hat{\omega}\frac{t^2}{2!} + \hat{\omega}^2\frac{t^3}{3!} + \dots\right) v + e^{\hat{\omega}t}\hat{\omega}v - e^{\hat{\omega}t}\hat{\omega}v$$

The last term $-e^{\hat{\omega}t}\hat{\omega}v$ is already in the desired format – with $t = 1/\|\omega\|$ – and part of the entry of interest.

For $+e^{\hat{\omega}t}\hat{\omega}v$ let us note that we can rewrite it as

$$\begin{aligned} e^{\hat{\omega}t}\hat{\omega}v &= \left(\hat{\omega} + \hat{\omega}^2 t + \hat{\omega}\frac{(\hat{\omega}t)^2}{2!} + \hat{\omega}\frac{(\hat{\omega}t)^3}{3!} + \dots\right) v \\ &= \hat{\omega}v + \hat{\omega}^2\left(\frac{t}{1!} + \hat{\omega}\frac{t^2}{2!} + \hat{\omega}^2\frac{t^3}{3!} + \dots\right) v \\ &= \hat{\omega}v + \left(\omega\omega^\top - I\right)\left(\frac{t}{1!} + \hat{\omega}\frac{t^2}{2!} + \hat{\omega}^2\frac{t^3}{3!} + \dots\right) v \end{aligned}$$

The first term in the latter expression is as desired. The part corresponding to the factor $-I$ negates exactly the series we had before adding zero.

All what remains is the term

$$\omega\omega^\top\left(\frac{t}{1!} + \hat{\omega}\frac{t^2}{2!} + \hat{\omega}^2\frac{t^3}{3!} + \dots\right) v$$

Since

$$\omega\omega^\top\hat{\omega} \;=\; \omega\left(\hat{\omega}^\top\omega\right) \;=\; \omega\left(-\underbrace{\hat{\omega}\omega}_{=0}\right)$$

we have that the remaing term reduces to the very first summand of the series, i.e. it reduces to $\omega\omega^\top t$. By letting again $t = 1/\|\omega\|$ we obtain the last desired part of the open entry.