Mathematical Foundations of Computer Vision

Example Solutions – Assignment 5

Solution of Exercise No. 1

Is $S = \{v_1, v_2, v_3\}$ *with*

$$v_1 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \quad \textit{and} \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

a generating system of $\mathbb{R}^3$*? Give a verbal reasoning of what you compute.*

The basis of $\mathbb{R}^3$ is given by $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$. Can we find λ_1, λ_2 and λ_3 such that

$$\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = \vec{e}_i$$

for $i = 1, 2, 3$? We have a look at

$$\underbrace{\begin{pmatrix} 1 & -2 & 1 \\ 5 & -1 & 3 \\ 4 & 1 & 2 \end{pmatrix}}_{=:A} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \vec{e}_i.$$

A solution λ exists if and only if the matrix A is regular. It is then given by

$$\lambda = A^{-1} \cdot \vec{e}_i.$$

Here:

$$det(A) = det \begin{pmatrix} -1 & 3 \\ 1 & 2 \end{pmatrix} + 2 \cdot det \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix} + det \begin{pmatrix} 5 & -1 \\ 4 & 1 \end{pmatrix} = 0.$$

Thus the matrix A is not regular and the given set S is therefore not a generating system of $\mathbb{R}^3$.

Solution of Exercise No. 2

Determine a basis for the space of solutions and its dimension for

$$\begin{array}{rcrcrcl} 2x_1 & + & x_2 & + & 3x_3 & = & 0 \\ x_1 & & & + & 5x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 0 \end{array}$$

The system is given by

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 0 & 5 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 7 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 1 & 3 \\ 0 & -1 & 7 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

A basis for the space of solutions is $\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$ and its dimension is by definition 0.

Is the following set $S = \{u_1, u_2, u_3\}$ *a generating system or a basis or nothing from these two, of the* $\mathbb{R}^2$*?*

$$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad \textit{and} \quad u_3 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

The set is a generating system of the $\mathbb{R}^2$ because

$$u_1 - \frac{2}{3} u_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \wedge \ \frac{1}{3} u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The set is not a basis of $\mathbb{R}^2$ because its members are not linearly independent. It holds

$$2u_1 - u_2 + u_3 = 0.$$

Solution of Exercise No. 3

(a) Let

$$Bx = b \quad \textit{with} \quad B = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Determine a basis of the kernel of B.

The kernel of B is the space of solutions of the homogeneous system $Bx = 0$.

$$\begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow x = \alpha \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

A basis of the kernel of B is given by $\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$.

Determine a particular solution x_0 of $Bx = b$.

$$Bx = b \Leftrightarrow \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 1-\alpha \end{pmatrix}$$

A particular solution x_0 of $Bx = b$ is given by $x_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Give a geometrical interpretation of kernel, x_0 and x.

The geometrical interpretations are in the setting abovebecome obvious when plotting kernel, x_0 and x in the domain of $x - y$-coordinates:

- The kernel is a mapping that always includes the zero point, i.e. the origin. In our case it is a linear function with slope -1 in the $x - y$-domain.
- The particular solution x_0 is a point which is a member of the set of all possible solutions of the underlying system.
- The set of solutions x is given by translating the kernel in such a way that the particular solution is included. In our case we obtain a linear function that runs through the point x_0.

(b) Now, let

$$C = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

Compute bases for row space and kernel of C.

We compute the kernel of C:

$$Cx = \vec{0} \Leftrightarrow \begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{pmatrix} x = \vec{0} \Leftrightarrow \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} x = \vec{0}.$$

Set $x_4 = \alpha$, $x_3 = \beta$. Then

$$x_2 = 3x_3 - 2x_4 = 3\beta - 2\alpha$$
$$x_1 = -2x_2 + x_3 - 2x_4 = -5\beta + 2\alpha.$$

Thus a basis of the kernel of C is

$$\left\{ \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

We compute a basis of the row space of C:

$$\begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 & 2 & -1 & 2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

A basis of the row space of C is $\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ 2 \end{pmatrix} \right\}$.

(c) Verify at hand of C: The kernel and the row space of a matrix are orthogonal complements.

We check if

$$\begin{pmatrix} 2\alpha - 5\beta \\ -2\alpha + 3\beta \\ \beta \\ \alpha \end{pmatrix} \perp \begin{pmatrix} \gamma \\ 2\gamma + \delta \\ -\gamma - 3\delta \\ 2\gamma + 2\delta \end{pmatrix}$$

holds for all $\alpha, \beta \in \mathbb{R}$. Indeed the kernel and row space of C are orthogonal, because

$$(2\alpha - 5\beta) \cdot (\gamma) + (-2\alpha + 3\beta) \cdot (2\gamma + \delta) + \beta \cdot (-\gamma - 3\delta) + \alpha \cdot (2\gamma + 2\delta) = 0$$

is fulfilled.

(d) Prove $\operatorname{rank}(A) = \operatorname{rank}(A^\top)$.

Let $k := \max(m, n)$ where the latter define the size of a given matrix A. Then it is clear by definition that

$$\dim_C A + \dim_K A = k$$

where the lower indices denote the column space and the kernel, respectively.

By part (b) of this exercise, we can also infer that

$$\dim_R A + \dim_K A = k$$

where the lower index R denotes the row space. Since

$$\dim_R A = \dim_C A^\top$$

the assertion follows.

Solution of Exercise No. 4

Determine rank and defect, plus verify the Dimension Theorem for

$$D = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

We consider the issues of importance:

- rank of D:

$$rank(D) = rank \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} = rank \begin{pmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1$$

- defect of D:

$$Dx = 0 \Leftrightarrow \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow x = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ with } \alpha, \beta \in \mathbb{R}$$

  Thus the defect of D is 2.

- The dimension theorem is fulfilled:

$$rank(A) + def(A) = n \;\Leftrightarrow\; 1 + 2 = 3.$$

Solution of Exercise No. 5

(a) Determine for the following matrices the eigenvalues, their algebraic multiplicities, and the dimension of the associated eigenspaces:

$$E_1 = I \in \mathbb{R}^{n \times n} \;, \quad E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \;, \quad E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

- $E_1 = I \in \mathbb{R}^{n \times n}$
  - Eigenvalues: Compute zeroes of the characteristic polynomial

$$det(\lambda I - E_1) = det(\lambda I - I) = det((\lambda - 1)I) = (\lambda - 1)^n det(I) = (\lambda - 1)^n \stackrel{!}{=} 0$$
$$\Rightarrow \lambda_1 = \ldots = \lambda_n = 1$$

  - algebraic multiplicity of $\lambda = 1$: n
  - dimension of the associated eigenspaces:
    * for $\lambda = 1$:

$$(\lambda I - E_1)x = 0 \;\Leftrightarrow\; 0x = 0$$
$$\Rightarrow \text{eigenspace}: \; \{\vec{e}_1, \ldots, \vec{e}_n\}$$

      $\Rightarrow$ the dimension of the associated eigenspace is n
      $\Rightarrow$ algebraic and geometric multiplicity is the same

- $E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$
  - eigenvalues:

$$det(\lambda I - E_2) = det \begin{pmatrix} \lambda - 1 & -1 \\ 0 & \lambda \end{pmatrix} = (\lambda - 1) \cdot \lambda \stackrel{!}{=} 0$$
$$\Rightarrow \lambda_1 = 1, \lambda_2 = 0.$$

- algebraic multiplicity of $\lambda = 1$: 1

  algebraic multiplicity of $\lambda = 0$: 1
- dimension of the associated eigenspaces:
  - for $\lambda = 1$:

$$(1I - E_2)x = 0 \;\Leftrightarrow\; \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \;\Leftrightarrow\; \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{eigenspace} : \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

    $\Rightarrow$ the dimension of the associated eigenspace is 1
  - for $\lambda = 0$:

$$(0I - E_2)x = 0 \;\Leftrightarrow\; \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{eigenspace} : \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

    $\Rightarrow$ the dimension of the associated eigenspace is 1

- $E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$
  - eigenvalues:

$$det(\lambda I - E_3) = det \begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix} = \lambda^2 \overset{!}{=} 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = 0$$

  - algebraic multiplicity of $\lambda = 0$: 2
  - dimension of the associated eigenspaces:
    - for $\lambda = 0$:

$$(0I - E_3)x = 0 \;\Leftrightarrow\; \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \text{eigenspace} : \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

    $\Rightarrow$ the dimension of the associated eigenspace is 1

What can you learn from these examples about the relation between regularity of a matrix and the dimension of its eigenspace?

- The total dimension of eigenspaces of a $n \times n$-matrix must not be equal to n.
- The total dimension of the eigenspaces of a $n \times n$-matrix does not depend on the regularity of the matrix (as E_2 and E_3 are not regular).

(b) Let

$$F_1 = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$$

Verify that F_1 and F_2 have the same eigenvalues with identical algebraic multiplicities.

- The characteristic polynomial of F_1:

$$det(\lambda I - F_1) = det \begin{pmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{pmatrix} = \lambda \cdot (\lambda - 2) \cdot (\lambda - 3) + 2(\lambda - 2)$$
$$= \lambda^3 - 5\lambda^2 + 8\lambda - 4.$$

- The characteristic polynomial of F_2:

$$det(\lambda I - F_2) = det \begin{pmatrix} \lambda - 1 & 0 & 0 \\ -1 & \lambda - 2 & 0 \\ 3 & -5 & \lambda - 2 \end{pmatrix} = (\lambda - 1) \cdot (\lambda - 2) \cdot (\lambda - 2)$$
$$= (\lambda - 1) \cdot (\lambda^2 - 4\lambda + 4)$$
$$= \lambda^3 - 5\lambda^2 + 8\lambda - 4.$$

- The characteristic polynomial of F_1 and F_2 are identical. Thus both matrices have the same eigenvalues with the same algebraic multiplicities.

- The zero crossings of both characteristic polynomials are given by:

$$\lambda_1 = 1, \lambda_2 = \lambda_3 = 2.$$

- algebraic multiplicity of $\lambda = 1$: 1

  algebraic multiplicity of $\lambda = 2$: 2.

Determine for F_1 and F_2 the bases of the eigenspaces.

For the matrix F_1:

- for $\lambda = 1$:

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow x = \alpha \cdot \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \text{ mit } \alpha \neq 0.$$

  associated eigenspace: $\left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \right\}$

- for $\lambda = 2$:

$$\begin{pmatrix} 2 & 0 & 2 \\ -1 & 0 & -1 \\ -1 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

  associated eigenspace: $\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

For the matrix F_2:

- for $\lambda = 1$:

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 3 & -5 & -1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

associated eigenspace: $\left\{ \begin{pmatrix} -1 \\ 1 \\ -8 \end{pmatrix} \right\}$

- for $\lambda = 2$:

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 3 & -5 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & -5 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

associated eigenspace: $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

Are they diagonalizable? Give a reasoning.

- The matrix F_1 is diagonalizable, because
  - for all eigenvalues, the geometric multiplicity is identical to the algebraic multiplicity
  - F_1 has n linearly independent eigenvectors.
- The matrix F_2 is not diagonalizable, because
  - for the eigenvalue $\lambda = 2$ the algebraic multiplicity is 2 and thus unequal to the geometric multiplicity, which is 1.

(c) Let

$$G = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

Compute the eigenvalues of G.

- characteristic polynomial of G:

$$\begin{aligned} det(\lambda I - G) &= det \begin{pmatrix} \lambda - 4 & 0 & -1 \\ -2 & \lambda - 3 & -2 \\ -1 & 0 & \lambda - 4 \end{pmatrix} \\ &= (\lambda - 4) \cdot (\lambda - 3) \cdot (\lambda - 4) - (\lambda - 3) \\ &= (\lambda - 3) \cdot \big((\lambda - 4)^2 - 1\big) \\ &= (\lambda - 3) \cdot \big(\lambda^2 - 8\lambda + 15\big) \\ &= (\lambda - 3) \cdot (\lambda - 3) \cdot (\lambda - 5). \end{aligned}$$

- eigenvalues of G:

  $\Rightarrow \lambda_1 = \lambda_2 = 3, \lambda_3 = 5$

For each eigenvalue λ, compute rank and defect of $\lambda I - G$.

For $\lambda = 3$:

- rank:

$$rank(3I-\lambda) = rank \begin{pmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{pmatrix} = rank \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 1$$

- defect:

$$def(3I-\lambda) = def \begin{pmatrix} -1 & 0 & -1 \\ -2 & 0 & -2 \\ -1 & 0 & -1 \end{pmatrix} = def \begin{pmatrix} -1 & 0 & 0 \\ -2 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} = 2.$$

For $\lambda = 5$:

- rank:

$$rank(5I-G) = rank \begin{pmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{pmatrix} = rank \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} = rank \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} = 2.$$

- defect:

$$def(5I-G) = def \begin{pmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ -1 & 0 & 1 \end{pmatrix} = def \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ -1 & 0 & 0 \end{pmatrix} = def \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} = 1.$$

What can you infer by the result?

One can infer that rank and defect of $G - \lambda I$ can be a hint concerning the algebraic and geometric multiplicity of an eigenvalue λ.

Is G diagonalizable? Give a reasoning.

Since the rank of $G - \lambda I$ is one for $\lambda = 3$ which has algebraic multiplicity of 2, also the geometric multiplicity will be 2: As by the constituting equation for the rank of $G - 3I$, one sees that the defect $def(G - 3I)$ gives us the number of free parameters in the solution of $G - 3I = \vec{0}$, i.e. it is the dimension of the kernel. Therefore G has a full set of eigenvectors and can be diagonalized.