Mathematical Foundations of Computer Vision

Example Solutions – Assignment 3

Solution of Exercise No. 1

(a) For $R = (r_{ij}) \in SO(3)$, prove by using Cramer's rule that

$$\begin{aligned} r_{11} &= r_{22}r_{33} - r_{23}r_{32} \\ r_{22} &= r_{11}r_{33} - r_{13}r_{31} \\ r_{33} &= r_{11}r_{22} - r_{21}r_{12} \end{aligned}$$

Using Cramer's rule one obtains

$$R^{-1} = \frac{1}{det(R)} \cdot \begin{pmatrix} +det\begin{pmatrix} r_{22} & r_{23} \\ r_{32} & r_{33} \end{pmatrix} & -det\begin{pmatrix} r_{12} & r_{13} \\ r_{32} & r_{33} \end{pmatrix} & +det\begin{pmatrix} r_{12} & r_{13} \\ r_{22} & r_{23} \end{pmatrix} \\ -det\begin{pmatrix} r_{21} & r_{23} \\ r_{31} & r_{33} \end{pmatrix} & +det\begin{pmatrix} r_{11} & r_{13} \\ r_{31} & r_{33} \end{pmatrix} & -det\begin{pmatrix} r_{11} & r_{13} \\ r_{21} & r_{23} \end{pmatrix} \\ +det\begin{pmatrix} r_{21} & r_{22} \\ r_{31} & r_{32} \end{pmatrix} & -det\begin{pmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{pmatrix} & +det\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} \end{pmatrix}$$

Since $R \in SO(3)$, we obtain $det(R) = +1$ and $R^{-1} = R^T$ holds. Therefore

$$R^T = \begin{pmatrix} r_{11} & r_{21} & r_{31} \\ r_{12} & r_{22} & r_{32} \\ r_{13} & r_{23} & r_{33} \end{pmatrix} = \begin{pmatrix} r_{22}r_{33} - r_{23}r_{32} & -det\begin{pmatrix} r_{12} & r_{13} \\ r_{32} & r_{33} \end{pmatrix} & det\begin{pmatrix} r_{12} & r_{13} \\ r_{22} & r_{23} \end{pmatrix} \\ -det\begin{pmatrix} r_{21} & r_{23} \\ r_{31} & r_{33} \end{pmatrix} & r_{11}r_{33} - r_{13}r_{31} & -det\begin{pmatrix} r_{11} & r_{13} \\ r_{21} & r_{23} \end{pmatrix} \\ det\begin{pmatrix} r_{21} & r_{22} \\ r_{31} & r_{32} \end{pmatrix} & -det\begin{pmatrix} r_{11} & r_{12} \\ r_{31} & r_{32} \end{pmatrix} & r_{11}r_{22} - r_{12}r_{21} \end{pmatrix}$$

Thus $r_{11} = r_{22}r_{33} - r_{23}r_{32}$, $r_{22} = r_{11}r_{33} - r_{13}r_{31}$ and $r_{33} = r_{11}r_{22} - r_{12}r_{21}$ is fulfilled.

(b) Prove that

1. *similar matrices A and B have the same characteristic polynomials.*
2. *the geometric multiplicity of the eigenvalues of A and B is the same.*

We first show that $det(\lambda I - A) = det(\lambda I - B)$ holds for similar matrices A and $B = U^{-1}AU$.

$$\begin{aligned} det(\lambda I - B) &= det(\lambda I - U^{-1}AU) \\ &= det(\lambda I \cdot U^{-1}U - U^{-1}AU) \\ &= det(U^{-1}(\lambda I)U - U^{-1}AU) \\ &= det(U^{-1}(\lambda I - A)U) \\ &= det(U^{-1}) \cdot det(\lambda I - A) \cdot det(U) \\ &= det(U^{-1}) \cdot det(U) \cdot det(\lambda I - A) \\ &= det(U^{-1}U) \cdot det(\lambda I - A) \\ &= det(I) \cdot det(\lambda I - A) \\ &= det(\lambda I - A). \end{aligned}$$

Thus they have the same characteristic polynomials.

It is evident that the algebraic multiplicity is equal. We now show that also the geometric multiplicity is the same. To this end, we have to show that the dimensions of eigenspaces is the same.

We first show that if x is an eigenvector of B, then Ux is an eigenvector of A corresponding to the eigenvalue λ.

$$\begin{aligned}
x \text{ is an eigenvector } of B &\Rightarrow Bx = \lambda x \\
&\Rightarrow (U^{-1}AU)x = \lambda x \\
&\Rightarrow U^{-1}AUx = \lambda x \\
&\Rightarrow AUx = U\lambda x \\
&\Rightarrow A(Ux) = \lambda(Ux).
\end{aligned}$$

U is a regular matrix. From $B = U^{-1}AU$ we obtain $A = UBU^{-1}$. In the next step we show that if x is an eigenvector of A, then $U^{-1}x$ is an eigenvector of B corresponding to the same eigenvalue.

$$\begin{aligned}
x \text{ is an eigenvector } of A &\Rightarrow Ax = \lambda x \\
&\Rightarrow UBU^{-1}x = \lambda x \\
&\Rightarrow BU^{-1}x = U^{-1}\lambda x \\
&\Rightarrow B(U^{-1}x) = \lambda(U^{-1}x).
\end{aligned}$$

The geometric multiplicity is the dimension of the associated eigenspace. Since U maps the eigenspace of B to the eigenspace of A and U^{-1} maps the eigenspace of A to the eigenspace of B, the dimensions of the eigenspaces have to be equal.

(c) Given is the matrix

$$A := \frac{1}{9}\begin{pmatrix} 0 & -1 & -2 \\ -1 & 0 & -2 \\ -2 & -2 & -3 \end{pmatrix} \tag{1}$$

Compute all eigenvalues of A.

First we compute the characteristic polynomial of A:

$$\begin{aligned}
det(\lambda I - A) &= det\left(\frac{1}{9}\begin{pmatrix} 9\lambda & 1 & 2 \\ 1 & 9\lambda & 2 \\ 2 & 2 & 9\lambda + 3 \end{pmatrix}\right) \\
&= \frac{1}{729}\left[9\lambda \cdot det\begin{pmatrix} 9\lambda & 2 \\ 2 & 9\lambda + 3 \end{pmatrix} - det\begin{pmatrix} 1 & 2 \\ 2 & 9\lambda + 3 \end{pmatrix} + 2 \cdot det\begin{pmatrix} 1 & 2 \\ 9\lambda & 2 \end{pmatrix}\right] \\
&= \frac{1}{729}\left[9\lambda \cdot (9\lambda \cdot (9\lambda + 3) - 4) - (9\lambda + 3 - 4) + 2 \cdot (2 - 18\lambda)\right] \\
&= \frac{1}{729}\left[9\lambda \cdot (81\lambda^2 + 27\lambda - 4) - 9\lambda + 1 + 4 - 36\lambda\right] \\
&= \frac{1}{729}\left[729\lambda^3 + 243\lambda^2 - 36\lambda - 9\lambda + 5 - 36\lambda\right] \\
&= \frac{1}{729}\left[729\lambda^3 + 243\lambda^2 - 81\lambda + 5\right].
\end{aligned}$$

Its zero crossings are

$$\begin{aligned}
& det(\lambda I - A) = 0 \\
\Leftrightarrow\ & 729\lambda^3 + 243\lambda^2 - 81\lambda + 5 = 0 \\
\Leftrightarrow\ & \lambda_{1,2} = \frac{1}{9}, \lambda_3 = -\frac{5}{9}.
\end{aligned}$$

Determine a basis for the resulting eigenspaces.

We compute the solution of $(\lambda I - A)x = 0$ for $\lambda = \frac{1}{9}$ and $\lambda = -\frac{5}{9}$.
With $\lambda = -\frac{5}{9}$:

$$\begin{pmatrix} -\frac{5}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{1}{9} & -\frac{5}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & -\frac{5}{9} + \frac{3}{9} \end{pmatrix} x = \frac{1}{9} \begin{pmatrix} -5 & 1 & 2 \\ 1 & -5 & 2 \\ 2 & 2 & -2 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} -5 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let $x_3 := \gamma$. Then $x_2 = \frac{1}{2}\gamma$ and $x_1 = \frac{1}{2}\gamma$. Thus for all $\gamma \neq 0$ the vector $x = \gamma \cdot \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$ is an eigenvector.

We will be interested in a normalized version of the latter vector: $v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$.

With $\lambda = \frac{1}{9}$:

$$\begin{pmatrix} \frac{1}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} + \frac{3}{9} \end{pmatrix} x = \frac{1}{9} \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Let $x_1 := \alpha$ and $x_2 := \beta$. Then

$$\begin{aligned} x_1 + x_2 + 2x_3 = 0 \;&\Leftrightarrow\; \alpha + \beta = -2x_3 \\ &\Leftrightarrow\; x_3 = -\frac{1}{2}\alpha - \frac{1}{2}\beta. \end{aligned}$$

If $x \neq 0$, then $x = \alpha \cdot \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix}$ is an eigenvector.

Finally

$$\left\{ \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2} \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -\frac{1}{2} \end{pmatrix} \right\}$$

is a basis for the resulting eigenspaces.

For later use, we note that the choice $\alpha = \beta = -1$ and the multiplication with a normalization factor gives the particular eigenvector $v_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$. It holds $\langle v_1, v_2 = 0$.

By $v_1 \times v_2 =: v_3 = \frac{1}{\sqrt{18}} \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix}$ we obtain another orthonormal eigenvector, equivalent to $\alpha = 3$, $\beta = -3$ and multiplication with a normalization weight.

Determine an orthogonal matrix U such that $\Lambda = U^\top A U$ is of diagonal form.

The sought matrix U can obviously be made from the columns of the orthonormal eigenvectors above, i.e. $U = [v_1, v_2, v_3]$.

Which transformation steps are described by the factors in the mapping $u \mapsto U\Lambda U^\top$?

We consider a vector x, given in the natural basis, and analyse what happens when calculating $Ax = U\Lambda U^T x$.

The multiplication with $U^T = U^{-1}$ transforms it into a new basis consisting of orthonormal eigenvectors of A. Applying the diagonal matrix has no effect on the basis, it describes a simple transformation in that new basis. And finally multiplying with U transforms the resulting vector back into the natural basis.

Exercise No. 2 – Treasure of the Indian Ocean

(a) Show the derivation of

$$\text{(a)} \qquad \cos\phi \;=\; \frac{1}{2}\,(\mathit{trace}(R) - 1)$$

Using $tr(A+B) = tr(A) + tr(B)$ and $tr(\alpha A) = \alpha tr(A)$, we obtain:

$$\begin{aligned} tr(R) &= tr\left(I\cos\varphi + \hat{v}\sin\varphi + vv^T(1-\cos\varphi)\right) \\ &= tr(I\cos\varphi) + tr(\hat{v}\sin\varphi) + tr(vv^T(1-\cos\varphi)) \\ &= \cos(\varphi)\cdot tr(I) + \sin(\varphi)\cdot tr(\hat{v}) + (1-\cos\varphi)\cdot tr(vv^T). \end{aligned}$$

Since $\hat{v}$ is a specific skew symmetric matrix with zero entries on the diagonal, it holds $tr(\hat{v}) = 0$.
With $\|v\| = 1$, we obtain

$$\begin{aligned} tr(vv^T) &= tr\begin{pmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_1v_2 & v_2^2 & v_2v_3 \\ v_1v_3 & v_2v_3 & v_3^2 \end{pmatrix} \\ &= v_1^2 + v_2^2 + v_3^2 \\ &= \|v\|_2^2 \\ &= 1. \end{aligned}$$

Finally

$$\begin{aligned} tr(R) &= \cos(\varphi)\cdot tr(I) + \sin(\varphi)\cdot tr(\hat{v}) + (1-\cos(\varphi))\cdot tr(vv^T) \\ &= \cos(\varphi)\cdot 3 + 1 - \cos(\varphi) \\ &= 2\cos(\varphi) + 1. \end{aligned}$$

This is equivalent to $tr(R) - 1 = 2\cos(\varphi)$. Multiplying with $\frac{1}{2}$ finishes the proof.

(b) Show the derivation of

$$\text{(b)} \qquad \hat{v} \;=\; \frac{1}{2\sin\phi}\left(R - R^\top\right)$$

We compute the transpose of R:

$$\begin{aligned} R^T &= \left(I\cos\varphi + \hat{v}\sin\varphi + vv^T(1-\cos\varphi)\right)^T \\ &= \cos(\varphi)\cdot I^T + \sin(\varphi)\cdot\hat{v}^T + (1-\cos(\varphi))(vv^T)^T \\ &= \cos(\varphi)\cdot I + \sin(\varphi)\hat{v}^T + (1-\cos(\varphi))(vv^T) \end{aligned}$$

Then

$$\frac{1}{2\sin(\varphi)}(R-R^T) = \frac{1}{2\sin(\varphi)}(\hat{v}\sin(\varphi) - \hat{v}^T\sin(\varphi))$$
$$= \frac{1}{2}(\hat{v} - \hat{v}^T).$$

We have to show that

$$\hat{v} = \frac{1}{2}(\hat{v} - \hat{v}^T) \;\Leftrightarrow\; \frac{1}{2}\hat{v} = -\frac{1}{2}\hat{v}^T \;\Leftrightarrow\; \hat{v} = -\hat{v}^T.$$

holds. As

$$\hat{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

holds, this property is obviously fulfilled.

Exercise No. 3 – Twist it

(a) Show that D is in $SO(3)$.

We check if $D^T D = DD^T = I$ holds.

$$\begin{aligned}
D^T D &= \frac{1}{9}\begin{pmatrix} 8 & 4 & -1 \\ 1 & -4 & -8 \\ -4 & 7 & -4 \end{pmatrix} \cdot \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 4 & -4 & 7 \\ -1 & -8 & -4 \end{pmatrix} \\
&= \frac{1}{81}\begin{pmatrix} 8\cdot 8 + 4\cdot 4 + (-1)\cdot(-1) & 8\cdot 1 - 4\cdot 4 + 1\cdot 8 & -8\cdot 4 + 4\cdot 7 + 1\cdot 4 \\ 1\cdot 8 - 4\cdot 4 + 8\cdot 1 & 1\cdot 1 + 4\cdot 4 + 8\cdot 8 & -1\cdot 4 - 4\cdot 7 + 4\cdot 8 \\ -4\cdot 8 + 7\cdot 4 + 4\cdot 1 & -4\cdot 1 - 7\cdot 4 + 4\cdot 8 & 4\cdot 4 + 7\cdot 7 + 4\cdot 4 \end{pmatrix} \\
&= \frac{1}{81}\begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

and equivalently one obtains

$$DD^T = \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 4 & -4 & 7 \\ -1 & -8 & -4 \end{pmatrix} \cdot \frac{1}{9}\begin{pmatrix} 8 & 4 & -1 \\ 1 & -4 & -8 \\ -4 & 7 & -4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Further

$$\begin{aligned}
det(D) &= \left(\frac{1}{9}\right)^3 \cdot \left[8 \cdot det\begin{pmatrix} -4 & 7 \\ -8 & -4 \end{pmatrix} - 4 \cdot det\begin{pmatrix} 1 & -4 \\ -8 & -4 \end{pmatrix} + (-1) \cdot det\begin{pmatrix} 1 & -4 \\ -4 & 7 \end{pmatrix}\right] \\
&= \frac{1}{729} \cdot [8 \cdot (16 + 56) - 4 \cdot (-4 - 32) - (7 - 16)] \\
&= \frac{1}{729} \cdot [8 \cdot 72 + 4 \cdot 36 + 9] \\
&= \frac{1}{729} \cdot [576 + 144 + 9] \\
&= \frac{1}{729} \cdot 729 \\
&= 1.
\end{aligned}$$

Thus $D \in SO(3)$.

(b) Compute the rotation axis and normalise the result.

Using the formula in Exercise 2 and what we will compute in part (c) we obtain

$$\hat{v} = \frac{1}{2\sin(\varphi)}(R - R^T) \;=\; \frac{1}{2\sin(\frac{2}{3}\pi)} \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & 0 & \frac{5}{3} \\ \frac{1}{3} & -\frac{5}{3} & 0 \end{pmatrix} \;=\; \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & 0 & \frac{5}{3} \\ \frac{1}{3} & -\frac{5}{3} & 0 \end{pmatrix}.$$

With

$$v_1 = \hat{v}_{32}, v_2 = \hat{v}_{13}, v_3 = \hat{v}_{21}$$

the rotation axis is given by

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{3}}\frac{5}{3} \\ -\frac{1}{\sqrt{3}}\frac{1}{3} \\ \frac{1}{\sqrt{3}}\frac{1}{3} \end{pmatrix} = \frac{\sqrt{3}}{9} \begin{pmatrix} -5 \\ -1 \\ 1 \end{pmatrix}.$$

This vector is already normalized as

$$\left(\frac{\sqrt{3}}{9}\right)^2 \cdot ((-5)^2 + (-1)^2 + (1)^2) = \frac{3}{81} \cdot 27 = 1.$$

(c) Compute the angle of rotation.

We make use of the formula given in Exercise 2. Then

$$\cos(\varphi) = \frac{1}{2}(tr(D) - 1) \;\Leftrightarrow\; \cos(\varphi) = \frac{1}{2}(0 - 1) \;\Leftrightarrow\; \cos(\varphi) = -\frac{1}{2}.$$

With $\varphi \in [0°, 180°]$ this yields

$$\varphi = \frac{2}{3}\pi = 120°.$$

Exercise No. 4 – Choreography of the Twist

(a) Compute an orthonormal basis $\{w_1, w_2, w_3\}$ of the $\mathbb{R}^3$ with $w_1 || v$.

Let $w_1 := \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$. We choose

$$v_2 = \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T.$$

Then

$$w_2' = v_2 - \underbrace{\langle v_2, w_1 \rangle}_{=0} \cdot w_1 = v_2 \;\Rightarrow\; w_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

We choose

$$v_3 = \begin{pmatrix} 1 & 1 & 2 \end{pmatrix}^T.$$

Then

$$w_3' = v_3 - \underbrace{\langle v_3, w_1\rangle}_{=0} \cdot w_1 - \underbrace{\langle v_3, w_2\rangle}_{=0} \cdot w_2 = v_3 \;\Rightarrow\; w_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

The vectors w_1, w_2 and w_3 describe an orthonormal basis.

(b) Determine the matrix realising the rotation w.r.t. the basis $\{w_1, w_2, w_3\}$.

As the matrix shall describe a rotation of angle $\phi = \pi/2$ about the axis $v = w_1$, the desired rotation can be written as

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\pi/2 & -\sin\pi/2 \\ 0 & \sin\pi/2 & \cos\pi/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

(c) Compute the orthogonal matrix S for the basis transform $\{e_1, e_2, e_3\} \to \{w_1, w_2, w_3\}$.

Let $W := [w_1, w_2, w_3]$. Then with $I := [e_1, e_2, e_3]$, the matrix S of the basis transform reads $I = WS$, i.e. $S^{-1} = W = S^\top$ because of the orthogonality of S. This means

$$S = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} \end{pmatrix}$$

(d) Determine the matrix C describing the rotation in the canonical basis.

One can directly infer that $C = S^\top A S$, where A performs the actual rotation in the basis W. Plugging in the matrices from the previous parts yields

$$C = \frac{1}{3} \begin{pmatrix} 1 & 1+\sqrt{3} & -1+\sqrt{3} \\ 1-\sqrt{3} & 1 & -1-\sqrt{3} \\ -1-\sqrt{3} & -1+\sqrt{3} & 1 \end{pmatrix}$$