Mathematical Foundations of Computer Vision

Example Solution – Assignment 2

Solution of Exercise No. 1

(a) Transform the vector $a_1 := (3,3)^\top$ given in the basis B_1 to new coordinates b_1 w.r.t. B_2.

Let us first write down the matrices we might use:

$$B_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = B_1^{-1}$$
$$B_2 := \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \quad \Longrightarrow \quad B_2^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}$$

We compute $b_1 = Aa_1$, with $A = B_2^{-1}B_1$

$$A = B_2^{-1}B_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}$$

Therefore, we compute:

$$b_1 = Aa_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

(b) Transform the vector $b_2 := (2,-1)^\top$ given in the basis B_2 to new coordinates a_2 w.r.t. B_1.

We compute $a_2 = Ab_2$, with $A = B_1^{-1}B_2$

$$A = B_1^{-1}B_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

Therefore, we obtain:

$$b_1 = Aa_1 = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

(c) Compute $\|a_1 - a_2\|_2$.

$$||a_1 - a_2||_2 = \left\| \begin{pmatrix} 3 \\ 3 \end{pmatrix} - \begin{pmatrix} 5 \\ 1 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} -2 \\ 2 \end{pmatrix} \right\|_2 = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

(d) Compute $\|b_1 - b_2\|_{A^{-\top}A^{-1}}$ making use of the metric induced by the canonical inner product expressed in the basis B_2. Comment on your result: Did you expect it?

From the first exercise, we already know:

$$A = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix} \quad \Longrightarrow \quad A^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

Now, we compute in a straight forward way:

$$
\begin{aligned}
||b_1 - b_2||_{A^{-\top}A^{-1}} &= \left|\left|\begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}2\\-1\end{pmatrix}\right|\right|_{A^{-\top}A^{-1}} \\
&= \left|\left|\begin{pmatrix}0\\2\end{pmatrix}\right|\right|_{A^{-\top}A^{-1}} \\
&= \sqrt{< \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix} >_{A^{-\top}A^{-1}}} \\
&= \sqrt{\begin{pmatrix}0\\2\end{pmatrix}^\top \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}^{-\top} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{-1}{3} & \frac{2}{3} \end{pmatrix}^{-1} \begin{pmatrix}0\\2\end{pmatrix}} \\
&= \sqrt{\begin{pmatrix}0\\2\end{pmatrix}^\top \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix}0\\2\end{pmatrix}} \\
&= \sqrt{\begin{pmatrix}0\\2\end{pmatrix}^\top \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix}-2\\2\end{pmatrix}} \\
&= \sqrt{\begin{pmatrix}0\\2\end{pmatrix}^\top \begin{pmatrix}-2\\4\end{pmatrix}} \\
&= \sqrt{8} \\
&= 2\sqrt{2}
\end{aligned}
$$

We expected this result, as a coordinate transform of two points does not change their distance measured at hand of the canonical inner product (equivalent to the Euclidean norm).

Solution of Exercise No. 2

(a) Prove that the length of the vector $(x, y, z)^\top$ (in Cartesian coordinates) is given by

$$\sqrt{x^2 + y^2 + z^2} \tag{1}$$

by making use of the Theorem of Pythagoras.

First, we compute the length of the diagonal d in the x-y-plane (i.e. the length of the projection of the vector onto this plane).

Using the Theorem of Pythagoras, we get:

$$d^2 \ = x^2 + y^2 \quad \text{i.e.} \quad d \ = \ \sqrt{x^2 + y^2}$$

Then we compute the diagonal of the quader, taking into account also the third coordinate direction. This is identical to the length of the vector $a = (x, y, z)^\top$.

Using the Theorem of Pythagoras, we get:

$$||a||_2^2 \ = d^2 + z^2 \ = \ x^2 + y^2 + z^2 \ = \ \sqrt{x^2 + y^2 + z^2}$$

So the length of the vector is given by $\sqrt{x^2 + y^2 + z^2}$.

(b) Show that for any positive definite, symmetric matrix $S \in \mathbb{R}^{3\times 3}$, the mapping

$$\langle \cdot, \cdot \rangle_S \ : \ \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \quad \textit{with} \quad \langle u, v \rangle_S \ = \ u^\top S v \tag{2}$$

is a valid inner product on $\mathbb{R}^3$.

We need to elaborate on three issues:

- Linearity:

$$\begin{aligned}\langle u, \alpha v + \beta w\rangle_S &= u^\top S\,(\alpha v + \beta w) \\ &= \alpha u^\top S v + \beta u^\top S w \\ &= \alpha\langle u, v\rangle_S + \beta\langle u, w\rangle_S\end{aligned}$$

- Symmetry:

$$\langle u, v\rangle_S \;=\; \underbrace{u^\top S v}_{\in\mathbb{R}} \;=\; \underbrace{\left(u^\top S v\right)^\top}_{\in\mathbb{R}} \;=\; v^\top S^\top u \;\overset{S\text{ symmetric}}{=}\; v^\top S u \;=\; \langle v, u\rangle_S$$

- Non-negativity:

$$\langle v, v\rangle_S > 0$$

automatically holds for non-zero vectors v because S is assumed to be positive definite. Equality to zero can only hold if v is identical to the vector $\vec{0}$ by the same reason.

In the total of these properties, $\langle\cdot,\cdot\rangle_S$ is a valid inner product for S symmetric and positive definite.

Solution of Exercise No. 3

(a) Compute the angle between v_1 and v_2 as well as the length of the projection of v_1 onto v_2.

We know that the relation $< v_1, v_2 >= ||v_1|| \cdot ||v_2|| \cdot \cos(\sphericalangle(v_1, v_2))$ holds.

So we have to compute $\cos(\sphericalangle(v_1, v_2)) = \dfrac{< v_1, v_2 >}{||v_1|| \cdot ||v_2||}$:

$$\cos(\sphericalangle(v_1, v_2)) \;=\; \frac{< v_1, v_2 >}{||v_1|| \cdot ||v_2||} \;=\; \frac{< \begin{pmatrix}3\\0\\4\end{pmatrix}, \begin{pmatrix}-1\\2\\-2\end{pmatrix} >}{\left\|\begin{pmatrix}3\\0\\4\end{pmatrix}\right\| \cdot \left\|\begin{pmatrix}-1\\2\\-2\end{pmatrix}\right\|} \;=\; \frac{-3-8}{\sqrt{25}\cdot\sqrt{9}} \;=\; \frac{-11}{5\cdot 3} \;=\; \frac{-11}{15}$$

$$\Longrightarrow \alpha = \sphericalangle(v_1, v_2) \;=\; \cos^{-1}\left(\frac{-11}{15}\right) \;\approx\; 137,17$$

For the projection, we have to compute:

$$\begin{aligned} p \;&=< v_1, \frac{v_2}{||v_2||} > \cdot \frac{v_2}{||v_2||} \;=< \begin{pmatrix}3\\0\\4\end{pmatrix}, \frac{\begin{pmatrix}-1\\2\\-2\end{pmatrix}}{\left\|\begin{pmatrix}-1\\2\\-2\end{pmatrix}\right\|} > \cdot \frac{\begin{pmatrix}-1\\2\\-2\end{pmatrix}}{\left\|\begin{pmatrix}-1\\2\\-2\end{pmatrix}\right\|} \\ &=< \begin{pmatrix}3\\0\\4\end{pmatrix}, \frac{1}{3}\begin{pmatrix}-1\\2\\-2\end{pmatrix} > \cdot \frac{1}{3}\begin{pmatrix}-1\\2\\-2\end{pmatrix} \;=\; \frac{1}{9}\cdot < \begin{pmatrix}3\\0\\4\end{pmatrix}, \begin{pmatrix}-1\\2\\-2\end{pmatrix} > \cdot \begin{pmatrix}-1\\2\\-2\end{pmatrix} \\ &= \frac{1}{9}\cdot(-3-8)\cdot\begin{pmatrix}-1\\2\\-2\end{pmatrix} \;=\; \frac{-11}{9}\cdot\begin{pmatrix}-1\\2\\-2\end{pmatrix} \end{aligned}$$

The length of the projection is then given by $\|p\|$:

$$\|p\| \;=\; \frac{11}{9}\sqrt{(-1)^2 + 2^2 + (-2)^2} \;=\; \frac{11}{3}$$

(b) Compute a vector n orthogonal to v_1 and v_2 with $\|n\|_2 = 1$.

We compute:

$$n' = v_1 \times v_2 = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -8 \\ 2 \\ 6 \end{pmatrix}$$

Now, we have to normalize the vector:

$$n = \frac{n'}{||n'||} = \frac{\begin{pmatrix} -8 \\ 2 \\ 6 \end{pmatrix}}{\left\|\begin{pmatrix} -8 \\ 2 \\ 6 \end{pmatrix}\right\|} = \frac{1}{\sqrt{64+4+36}} \cdot \begin{pmatrix} -8 \\ 2 \\ 6 \end{pmatrix} = \frac{1}{\sqrt{104}} \cdot \begin{pmatrix} -8 \\ 2 \\ 6 \end{pmatrix} = \frac{1}{\sqrt{26}} \cdot \begin{pmatrix} -4 \\ 1 \\ 3 \end{pmatrix}$$

(c) Compute the area A of the parallelogram spanned by v_1 and v_2.

To compute the area A of the parallelogram spanned by v_1 and v_2, we have to compute:

$$A = \|v_1 \times v_2\| = \|n'\| = \left\|\begin{pmatrix} -8 \\ 2 \\ 6 \end{pmatrix}\right\| = \sqrt{64+4+36} = 2\sqrt{26}$$

Solution of Exercise No. 4

Now, let a point light source *be given at the point $P := (1,1,1)^\top$. Let the light shine onto a* triangle patch *determined by the vertices*

$$A := \begin{pmatrix} 1 \\ 4/3 \\ 1/3 \end{pmatrix}, \quad B := \begin{pmatrix} 1 \\ 3/2 \\ 1 \end{pmatrix} \quad \textit{and} \quad C := \begin{pmatrix} 5/2 \\ 1/2 \\ 0 \end{pmatrix} \tag{3}$$

Compute the area of the shadow of the triangle patch given by (A, B, C) on the plane $4x + 6y - 3z = 19$.

We write the plane onto which the shadow is cast as

$$e : \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix} \mathbf{x} - 19 = 0$$

For $A = (1, \frac{4}{3}, \frac{1}{3})^\top$, we compute the shadowpoint L_1. First, we compute the line through A and $P = (1,1,1)^\top$ as

$$g_{AP} : \mathbf{x} = P + \lambda_1(A - P) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix}$$

Then we compute the intersection of e and g_{AP}:

$$\begin{aligned} \begin{pmatrix} 4 \\ 6 \\ -3 \end{pmatrix} \left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{-2}{3} \end{pmatrix} \right) - 19 = 0 \\ \Leftrightarrow \quad 7 + 4\lambda_1 - 19 = 0 \\ \Leftrightarrow \quad \lambda_1 = 3 \end{aligned}$$

For the shadowpoint L_1, we thus get: $L_1 = (1, 2, -1)^\top$.

For $B = (1, \frac{3}{2}, 1)^\top$, we compute the shadowpoint L_2. First, we compute the line through B and P:

$$g_{BP} : \mathbf{x} \;=\; P + \lambda_2(B - P) \;=\; \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\\frac{1}{2}\\0\end{pmatrix}$$

Then we compute the intersection of e and g_{BP}:

$$\begin{aligned} \begin{pmatrix}4\\6\\-3\end{pmatrix}\left(\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_2\begin{pmatrix}0\\\frac{1}{2}\\0\end{pmatrix}\right) - 19 = 0 \\ \Leftrightarrow \quad 7 + 3\lambda_2 - 19 = 0 \\ \Leftrightarrow \quad \lambda_2 = 4 \end{aligned}$$

For the shadowpoint L_2, we get $L_2 = (1, 3, 1)^\top$.

For $C = (\frac{5}{2}, \frac{1}{2}, 0)^\top$, we compute the shadowpoint L_3. First, we compute the line through C and P:

$$g_{CP} : \mathbf{x} \;=\; P + \lambda_2(C - P) \;=\; \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_3 \begin{pmatrix}\frac{3}{2}\\\frac{-1}{2}\\-1\end{pmatrix}$$

Then we compute the intersection of e and g_{CP}:

$$\begin{aligned} \begin{pmatrix}4\\6\\-3\end{pmatrix}\left(\begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda_3\begin{pmatrix}\frac{3}{2}\\\frac{-1}{2}\\-1\end{pmatrix}\right) - 19 = 0 \\ \Leftrightarrow \quad 7 + 6\lambda_3 - 19 = 0 \\ \Leftrightarrow \quad \lambda_3 = 2 \end{aligned}$$

For the shadowpoint L_3, we get $L_3 = (4, 0, -1)^\top$.

In summary we have the three points $L_1 = (1, 2, -1)^\top$, $L_2 = (1, 3, 1)^\top$, $L_3 = (4, 0, -1)^\top$.

Now, we have to compute the area of the triangle:

$$A \;=\frac{1}{2} \cdot ||\vec{L_1L_2} \times \vec{L_1L_3}|| \;=\; \frac{1}{2} \cdot \left|\left|\begin{pmatrix}0\\1\\2\end{pmatrix} \times \begin{pmatrix}3\\-2\\0\end{pmatrix}\right|\right| \;=\; \frac{1}{2} \cdot \left|\left|\begin{pmatrix}4\\6\\-3\end{pmatrix}\right|\right| \;=\; \frac{\sqrt{61}}{2}$$