Mathematical Foundations of Computer Vision

Example Solutions – Assignment 1

In this assigment, we consider three candidates for bases of the good ol' 3-D Euclidean space:

$$B_1 := [e_1, e_2, e_3]\ , \quad B_2 := \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \quad \text{and} \quad B_3 := \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \tag{1}$$

Solution of Exercise No. 1

(a) Is B_2 a basis?

We check the rank of the matrix in order to observe if its column vectors are linearly independent (we use the fact that the rank w.r.t. the column vectors contained in B_2 will be the same as the rank w.r.t. its rows):

$$B_2 = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 1 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 11 \end{pmatrix}$$

$\Longrightarrow \quad \text{rank}(B_2) = 3$, so B_2 is a basis.

(b) Is B_3 a basis?

We employ the same procedure as in (a):

$$B_3 = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -6 \\ 0 & 1 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -6 \\ 0 & 1 & 3 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Longrightarrow \quad \text{rang}(B_3) = 2$, so B_3 is not a basis.

(c) For those matrices that do not represent a basis, state the subspace which is spanned by the vectors.

The subspace which is spanned by the column vectors in B_3 is $U = \{(1,3,0)^\top, (1,1,1)^\top\}$.

The reason for this choice is that these two vectors are obviously linearly independent, and one has to choose two such vectors because the rank of B_3 is two.

(d) Compute the volume contained in the parallelepipedon spanned by the column vectors of B_2.

Let $B_2 = [b_1, b_2, b_3]$. The volume is given by $|\det B_2|$:

$$\begin{aligned} |(b_1 \times b_2) \cdot b_3| &= |\det B_2| \\ &= |\begin{vmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix}| \\ &= |-2-9| \ = \ |-11| \ = \ 11 \end{aligned}$$

Solution of Exercise No. 2

(a) Determine the basis transformation from B_1 to B_2.

It holds: $B_1 = B_2 A \Longleftrightarrow A = B_2^{-1} B_1$
First, we compute B_2^{-1}:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 2 & -3 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 2 & -3 & 1 & 0 \\ 0 & 0 & 11 & -3 & 1 & 3 \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -33 & 0 & -27 & 9 & -6 \\ 0 & 0 & 1 & \frac{-3}{11} & \frac{1}{11} & \frac{3}{11} \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 33 & 0 & 0 & 6 & 9 & -6 \\ 0 & 1 & 0 & \frac{9}{11} & \frac{-3}{11} & \frac{2}{11} \\ 0 & 0 & 1 & \frac{-3}{11} & \frac{1}{11} & \frac{3}{11} \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{11} & \frac{3}{11} & \frac{-2}{11} \\ 0 & 1 & 0 & \frac{9}{11} & \frac{-3}{11} & \frac{2}{11} \\ 0 & 0 & 1 & \frac{-3}{11} & \frac{1}{11} & \frac{3}{11} \end{array}\right)$$

$$\Longrightarrow B_2^{-1} = \begin{pmatrix} \frac{2}{11} & \frac{3}{11} & \frac{-2}{11} \\ \frac{9}{11} & \frac{-3}{11} & \frac{2}{11} \\ \frac{-3}{11} & \frac{1}{11} & \frac{3}{11} \end{pmatrix}$$

Now we have to compute the basis transformation $A = B_2^{-1} B_1$.

$$A = \begin{pmatrix} \frac{2}{11} & \frac{3}{11} & \frac{-2}{11} \\ \frac{9}{11} & \frac{-3}{11} & \frac{2}{11} \\ \frac{-3}{11} & \frac{1}{11} & \frac{3}{11} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{11} & \frac{3}{11} & \frac{-2}{11} \\ \frac{9}{11} & \frac{-3}{11} & \frac{2}{11} \\ \frac{-3}{11} & \frac{1}{11} & \frac{3}{11} \end{pmatrix}$$

(b) Determine the basis transformation from B_2 to B_1.

It holds $B_2 = B_1 A$, i.e. $A = B_1^{-1} B_2$.

First, we compute B_1^{-1}. It holds $B_1 = I$, so that $B_1^{-1} = I$.

Now we have to compute the basis transformation $A = B_1^{-1} B_2 = I B_2 = B_2$:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

Solution of Exercise No. 3

(a) Transform the point

$$u := \begin{pmatrix} 22 \\ 55 \\ 11 \end{pmatrix} \quad \textit{in the Cartesian basis } B_1 \tag{2}$$

into new coordinates w.r.t. B_2.

It holds $B_1 u = B_2 A u$, so we have to compute $u' = Au$. Since we have for this exercise $A = B_2^{-1} B_1$, we know already the matrix A:

$$A = \begin{pmatrix} \frac{2}{11} & \frac{3}{11} & \frac{-2}{11} \\ \frac{9}{11} & \frac{-3}{11} & \frac{2}{11} \\ \frac{-3}{11} & \frac{1}{11} & \frac{3}{11} \end{pmatrix}$$

So, we compute $u' = Au$

$$u' = \begin{pmatrix} \frac{2}{11} & \frac{3}{11} & \frac{-2}{11} \\ \frac{9}{11} & \frac{-3}{11} & \frac{2}{11} \\ \frac{-3}{11} & \frac{1}{11} & \frac{3}{11} \end{pmatrix} \begin{pmatrix} 22 \\ 55 \\ 11 \end{pmatrix} = \begin{pmatrix} 4 + 15 - 2 \\ 18 - 15 + 2 \\ -6 + 5 + 3 \end{pmatrix} = \begin{pmatrix} 17 \\ 5 \\ 2 \end{pmatrix}$$

So the transformed point written in terms of coordinates w.r.t. B_2 is $u' = (17, 5, 2)^\top$.

(b) Transform the point

$$w := \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \textit{given in the basis } B_2 \tag{3}$$

to Cartesian coordinates.

It holds $B_2 w = B_1 A w$, so we have to compute $w' = Aw$ with $A = B_1^{-1} B_2$.

We already know the matrix A:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

So, we compute $w' = Aw$:

$$w' = \begin{pmatrix} 1 & 1 & 0 \\ 3 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1+2+0 \\ 3+0+4 \\ 0+2+6 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 8 \end{pmatrix}$$

So the transformed point written in terms of the coordinates w.r.t. B_1 is $w' = (3, 7, 8)^\top$.