Mathematical Foundations of Computer Vision

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Assignment 5 – Linear Algebra from Scratch (LFS)

Exercise No. 1 – Start the System

Let $S = \{v_1, v_2, \ldots, v_k\} \subset V$ and $W = \text{span}(S)$.

The set S is the *generating system* of W, and W is the *linear hull* of $v_1, v_2, \ldots, v_k$.

Is $S = \{v_1, v_2, v_3\}$ with

$$v_1 = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$$

a generating system of $\mathbb{R}^3$? Give a verbal reasoning of what you compute. **(2pts)**

Exercise No. 2 – Zero Space in the Matrix

A set S of vectors is a *basis* of a vector space V if the members of S (= S itself) are linearly independent, and if S is a generating system of V.

The *dimension* $\dim V$ of V is identical to the number of elements in a basis. For $V = \{\vec{0}\}$, the *zero space* (which is a vector space by itself!), we set $\dim V = 0$.

Determine a basis for the space of solutions and its dimension for

$$\begin{array}{ccccccc} 2x_1 & + & x_2 & + & 3x_3 & = & 0 \\ x_1 & & & + & 5x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 0 \end{array}$$

(2pts)

Is the following set $S = \{u_1, u_2, u_3\}$ a generating system or a basis or nothing from these two, of the $\mathbb{R}^2$?

$$u_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \quad \text{and} \quad u_3 = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$$

(2pts)

Exercise No. 3 – Run the Kernel

For a matrix $A \in \mathbb{R}^{m \times n}$

- the subspace of $\mathbb{R}^m$ spanned by its column vectors is its *column space*;
- the subspace of $\mathbb{R}^n$ spanned by its row vectors is its *row space*;
- the space of solutions of the homogeneous system $Ax = \vec{0}$ is the *kernel*.

One can show: The *general solution* x of $Ax = b$ is equal to a *(particular) solution* x_0 of $Ax = b$ plus the *general solution* $c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$ of $Ax = \vec{0}$:

$$x = x_0 + c_1 v_1 + c_2 v_2 + \ldots + c_k v_k$$

(a) Let

$$Bx = b \quad \text{with} \quad B = \begin{pmatrix} 1 & 1 \\ -2 & -2 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Determine a basis of the kernel of B, and a particular solution x_0 of $Bx = b$. Give a geometrical interpretation of kernel, x_0 and x. **(4pts)**

Now, let

$$C = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{pmatrix}$$

(b) Compute bases for row space and kernel of C. **(2pts)**

(c) Verify at hand of C: The kernel and the row space of a matrix are orthogonal complements. **(2pts)**

The dimension of column/row space of A is the *rank* of A and denoted $\operatorname{rank}(A)$, the dimension of the kernel is called *defect* of A, written as $\operatorname{def}(A)$.

(d) Prove $\operatorname{rank}(A) = \operatorname{rank}(A^\top)$. **(2pts)**

Exercise No. 4 – Dimensionalize the System

One can prove the

Dimension Theorem. For a matrix A with n columns holds $\operatorname{rank}(A) + \operatorname{def}(A) = n$.

Determine rank and defect, plus verify the Dimension Theorem for

$$D = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

(4pts)

Exercise No. 5 – Matrix Eigenheiten$^{\mathrm{TM}}$

For a quadratic matrix A holds: *(i)* The geometric multiplicity of an eigenvalue is not larger than its algebraic multiplicity, and *(ii)* A is *diagonalizable* (i.e. it is similar to a diagonal matrix) if and only if for *each* eigenvalue, the geometric multiplicity is identical to the algebraic multiplicity.

One can also show: For $A \in \mathbb{R}^{n\times n}$, A is diagonalizable if and only if n has n linearly independent eigenvectors.

(a) Determine for the following matrices the eigenvalues, their algebraic multiplicities, and the dimension of the associated eigenspaces:

$$E_1 = I \in \mathbb{R}^{n\times n}\ , \quad E_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\ , \quad E_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

What can you learn from these examples about the relation between regularity of a matrix and the dimension of its eigenspace? **(8pts)**

(b) Let

$$F_1 = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}\ , \quad F_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}$$

Verify that F_1 and F_2 have the same eigenvalues with identical algebraic multiplicities. Determine for F_1 and F_2 the bases of the eigenspaces. Are they diagonalizable? Give a reasoning. **(6pts)**

(b) Let

$$G = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix}$$

Compute the eigenvalues of G. For each eigenvalue λ, compute rank and defect of $\lambda I - G$. What can you infer by the result? Is G diagonalizable? Give a reasoning. **(6pts)**