Mathematical Foundations of Computer Vision

Michael Breuß

Released: 18.11.2011
Assigned to: Tutorial at 24.11.2011

Assignment 4 – Matrix Reloaded

Exercise No. 1 – Enter the Matrix

We consider a function $y = \varphi(x)$ with $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, and where φ is some transformation.

The *Jacobian matrix* of φ is defined as

$$\frac{\partial y}{\partial x} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \in \mathbb{R}^{m \times n} \tag{1}$$

(a) Let $y = Ax$ with $y \in \mathbb{R}^m$, $x \in \mathbb{R}^n$, and where $A = (a_{ij})$, $A \in \mathbb{R}^{m \times n}$, does not depend on x.

Prove or disprove: $\frac{\partial y}{\partial x} = A$ **(2pts)**

(b) Let $f = x^\top Ax$ be given where $f \in \mathbb{R}$, $x \in \mathbb{R}^n$, $A = (a_{ij})$, $A \in \mathbb{R}^{n \times n}$.

Compute $\frac{\partial f}{\partial x}$ for *(i)* A not symmetric, and for *(ii)* A symmetric. **(4pts)**

(c) Let $f(z) = y^\top(z)x(z)$ where $z \in \mathbb{R}^n$, $x(z) \in \mathbb{R}^n$, $y(z) \in \mathbb{R}^n$.

Compute $\frac{\partial f}{\partial z}$. **(2pts)**

(d) Let $\varphi(x) = \|x - v\|_2$, where $x, v \in \mathbb{R}^n$.

Compute $\frac{\partial \varphi}{\partial x}$. **(2pts)**

Exercise No. 2 – Differentiate the Matrix

Now, let $A = (a_{ij})$ be a $m \times n$ matrix with $a_{ij} = a_{ij}(t)$, $t \in \mathbb{R}$. Then

$$\frac{d}{dt}A(t) = \dot{A}(t) = \begin{pmatrix} \frac{da_{11}}{dt} & \cdots & \frac{da_{1n}}{dt} \\ \vdots & \ddots & \vdots \\ \frac{da_{m1}}{dt} & \cdots & \frac{da_{mn}}{dt} \end{pmatrix} \in \mathbb{R}^{m \times n} \tag{2}$$

(a) Let $B, C \in \mathbb{R}^{n \times n}$ with $B = (b_{ij})$, $b_{ij} = b_{ij}(t)$ and $C = (c_{ij})$, $c_{ij} = c_{ij}(t)$.

Let $BC = I$. Compute the equation resulting out of

$$\frac{d}{dt}[BC] = \frac{d}{dt}[I]$$

(4pts)

(b) Let $A = (a_{ij}) \in \mathbb{R}^{m \times n}$ be invertible, with $a_{ij} = a_{ij}(t)$.

Compute $\frac{d}{dt}\left[A^{-1}\right]$. **(4pts)**

Exercise No. 3 – Matrix Trinity

We consider again the *similarity* property.

We remember, that for given $A \in \mathbb{R}^{n\times n}$ there is a similar matrix Λ if we have an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$ with $A = Q\Lambda Q^\top$.

(a) Prove that the following implication holds:
If A can be made similar to a diagonal matrix Λ, **then** A is symmetric. **(4pts)**

We have made use already of such matrices Q composed of the eigenvectors of A. We give this some more basement:

(b) Prove that the following assertion holds:
For a symmetric matrix A, the eigenvalues are real. **(4pts)**

We supplement this by:

(c) Prove that the following assertion holds:
For a symmetric matrix A, the eigenvectors to different eigenvalues are orthogonal. **(4pts)**