Mathematical Foundations of Computer Vision

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Assignment 3 – Rigid Body-Workout

Exercise No. 1 – Slow Down Baby

We stretch our muscles via a few exercises upon important assertions of the mathematical *back to basics* tracklist.

(a) For $R = (r_{ij}) \in SO(3)$, prove by using Cramer's rule that

$$\begin{aligned} r_{11} &= r_{22}r_{33} - r_{23}r_{32} \\ r_{22} &= r_{11}r_{33} - r_{13}r_{31} \\ r_{33} &= r_{11}r_{22} - r_{21}r_{12} \end{aligned}$$

(2pts)

(b) Two quadratic matrices A and B are called *similar* if a regular matrix U exists, so that $B = U^{-1}AU$. The transform $A \mapsto U^{-1}AU$ is called *similarity transform*.
Let us also recall the definition of the *geometric multiplicity* of an eigenvalue: It is the dimension of the associated eigenspace.

Prove that

1. similar matrices A and B have the same characteristic polynomials. **(4pts)**
2. the geometric multiplicity of the eigenvalues of A and B is the same. **(4pts)**

(c) Given is the matrix

$$A := \frac{1}{9}\begin{pmatrix} 0 & -1 & -2 \\ -1 & 0 & -2 \\ -2 & -2 & -3 \end{pmatrix} \tag{1}$$

Compute all eigenvalues of A and determine a basis for the resulting eigenspaces. Determine an orthogonal matrix U such that $\Lambda = U^\top AU$ is of diagonal form. Which transformation steps are described by the factors in the mapping $u \mapsto U\Lambda U^\top$? **(6pts)**

Exercise No. 2 – Treasure of the Indian Ocean

We relax – making use of the *formula of Rodrigues*:

$$R = I\cos\phi + \hat{v}\sin\phi + vv^\top(1-\cos\phi) \tag{2}$$

We dive (in the ocean of math) for the following, precious expressions for the angle ϕ and the axis of rotation v from a given general rotation matrix $R \in \mathbb{R}^{3\times 3}$:

$$\text{(a)} \quad \cos\phi = \frac{1}{2}\left(\text{trace}(R) - 1\right)$$

$$\text{(b)} \quad \hat{v} = \frac{1}{2\sin\phi}\left(R - R^\top\right)$$

The task is to show the derivation of these formulae in detail. **(4+4pts)**

Exercise No. 3 – Twist it

Let the matrix

$$D := \frac{1}{9}\begin{pmatrix} 8 & 1 & -4 \\ 4 & -4 & 7 \\ -1 & -8 & -4 \end{pmatrix} \tag{3}$$

be given.

(a) Show that D is in $SO(3)$. **(2pts)**

(b) Compute the rotation axis and normalise the result. **(3pts)**

(c) Compute the angle of rotation. **(3pts)**

Exercise No. 4 – Choreography of the Twist

Our aim is to describe the rotation of the $\mathbb{R}^3$ about the axis $v = (1, 1, -1)^\top$ and the angle $\phi = \pi/2$.

(a) Compute an orthonormal basis $\{w_1, w_2, w_3\}$ of the $\mathbb{R}^3$ with $w_1 || v$. **(2pts)**

(b) Determine the matrix realising the rotation w.r.t. the basis $\{w_1, w_2, w_3\}$. **(2pts)**

(c) Compute the orthogonal matrix S for the basis transform $\{e_1, e_2, e_3\} \to \{w_1, w_2, w_3\}$. **(2pts)**

(d) Determine the matrix C describing the rotation in the canonical basis. **(2pts)**