Mathematical Foundations of Computer Vision

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Assigned to: Tutorial at 03.11.2011, **this tutorial will start 16:45**!

Assignment 2 – The Concrete Basement Sheet

Two pages of small exercises cementing the basics.

Exercise No. 1 – Plain math

In this exercise, we consider the 2-D Euclidean space together with the two bases

$$B_1 := [e_1, e_2] \quad \text{and} \quad B_2 := \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \tag{1}$$

(a) Transform the vector $a_1 := (3,3)^\top$ given in the basis B_1 to new coordinates b_1 w.r.t. B_2. **(1pt)**

(b) Transform the vector $b_2 := (2,-1)^\top$ given in the basis B_2 to new coordinates a_2 w.r.t. B_1. **(1pt)**

(c) Compute $\|a_1 - a_2\|_2$. **(2pts)**

(d) Compute $\|b_1 - b_2\|_{A^{-\top}A^{-1}}$ making use of the metric induced by the canonical inner product expressed in the basis B_2. Comment on your result: Did you expect it? **(4pts)**

Exercise No. 2 – Being innerly productive

We deal now with aspects of inner products:

(a) Prove that the length of the vector $(x,y,z)^\top$ (in Cartesian coordinates) is given by

$$\sqrt{x^2 + y^2 + z^2} \tag{2}$$

by making use of the Theorem of Pythagoras. **(3pts)**

(b) Show that for any positive definite, symmetric matrix $S \in \mathbb{R}^{3\times 3}$, the mapping

$$\langle \cdot, \cdot \rangle_S \;:\; \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \quad \text{with} \quad \langle u, v \rangle_S \;=\; u^\top S v \tag{3}$$

is a valid inner product on $\mathbb{R}^3$. **(3pts)**

Exercise No. 3 – Being crosswise productive

Let

$$v_1 \;:=\; \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} \quad \text{and} \quad v_2 \;:=\; \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix} \tag{4}$$

(a) Compute the angle between v_1 and v_2 as well as the length of the projection of v_1 onto v_2. **(2pts)**

(b) Compute a vector n orthogonal to v_1 and v_2 with $\|n\|_2 = 1$. **(2pts)**

(c) Compute the area A of the parallelogram spanned by v_1 and v_2. **(2pts)**

Exercise No. 4 – Plain stuff in 3-D

The *normal equation* of a 2-D plane P in 3-D reads as

$$\langle \vec{x} - \vec{a}, \vec{n} \rangle \; = \; 0 \tag{5}$$

where $\vec{n}$ is orthogonal to the plane and $\vec{a}$ contains the coordinates of a point in P. For $\|\vec{n}\|_2 = 1$, the formula (5) gives the *Hesse normal form*

$$\langle \vec{x}, \vec{n} \rangle - d \; = \; 0 \tag{6}$$

where $d \in \mathbb{R}$ is the distance between the origin and its closest point in P. In contrast, the *parameter form* of a plane is of the format

$$\vec{x} \; = \; \vec{x}_0 + \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 \quad \text{with} \quad \lambda_1, \lambda_2 \in \mathbb{R} \tag{7}$$

where $\vec{x}_0$ is a point on P, and where $\vec{v}_1, \vec{v}_2$ give the basis of the 2-D subspace containing $\vec{x}_0$.

Now, let a *point light source* be given at the point $p := (1,1,1)^\top$. Let the light shine onto a *triangle patch* determined by the vertices

$$A \; := \; \begin{pmatrix} 1 \\ 4/3 \\ 1/3 \end{pmatrix}, \quad B \; := \; \begin{pmatrix} 1 \\ 3/2 \\ 1 \end{pmatrix} \quad \text{and} \quad C \; := \; \begin{pmatrix} 5/2 \\ 1/2 \\ 0 \end{pmatrix} \tag{8}$$

Compute the area of the shadow of the triangle patch given by (A, B, C) on the plane $4x + 6y - 3z = 19$.

(10pts)