On Implicit Image Derivatives and Their Applications

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Outline

1. Introduction
   - Motivation
   - Current Approach
   - Previous Work
2. Background
   - Explicit vs. Implicit Methods
   - Taylor and Padé Approximations
   - Differentiation as a Linear Operator
3. From Explicit to Implicit Differentiation Schemes
   - Standard Explicit Schemes
   - From Explicit to Implicit Schemes
   - Advanced Implicit Schemes
4. Discussion

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The Importance of Image Derivatives

- differentiation: one of the most fundamental tasks of low level image processing
- used to detect edges and corners: perceptual building blocks
- necessary for a host of other image processing operations: e.g. smoothing, deblurring, segmentation

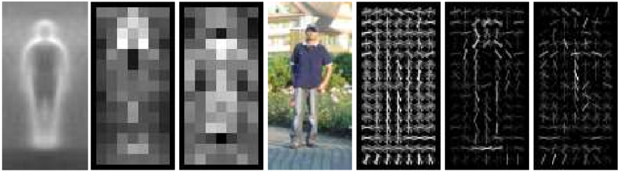

Figure: From Dalal & Triggs, 2005

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Contributions

- derivatives in image processing typically obtained using imprecise explicit schemes
- main contributions [2, 3]:
  1. establishing a link between implicit and explicit finite differences used for gradient estimation
  2. introducing new implicit differencing schemes and evaluating their properties
  3. attempting to demonstrate the usefulness and potential of implicit finite differencing schemes for image processing tasks

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Implicit Finite Differences

- common among numerical mathematicians and computational physicists
- seminal paper by Lele [1]: demonstrated superior performance of implicit finite difference schemes
- uses (among others):
  1. accurate numerical simulations of physical problems involving wave propagation phenomena
  2. modelling weather phenomena
  3. accurate visualisation of volumetric data

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Explicit vs. Implicit Methods

- categorisation of numerical schemes
    - explicit methods: dependent variable can be obtained directly from input variables
    - implicit methods: more complex relationship between variables, requires solving systems of linear equations

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Taylor Approximations

Definition (Taylor Approximant of Order k)

$$\begin{aligned} f(x+h) &= \sum_{i=0}^{k} \frac{f^{(i)}(x)}{i!}(h)^i + R_k(h) \\ &= f(x) + f'(x)(h) + \ldots + \frac{f^{(k)}(x)}{k!}(h)^k + R_k(h) \end{aligned} \quad (1)$$

- approximation of a function f, centered at x
- used to derive explicit finite difference schemes

Padé Approximations

Definition (Padé Approximant of Order m/n)

$$R(x) = \frac{\sum_{j=0}^{m} a_j x^j}{1 + \sum_{k=1}^{n} b_k x^k} = \frac{a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m}{1 + b_1 x + b_2 x^2 + \ldots + b_n x^n} \tag{2}$$

- approximation of a function by a rational function
- often more precise than the Taylor approximation
- later used to derive powerful implicit differentiation schemes

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Differentiation in the Frequency Domain

- differentiation is a linear operation, thus has interesting properties in the frequency domain
- in particular: $F[f^{n)}] = (j\omega)^n F[f](u), \qquad \text{with } \omega = 2\pi u$

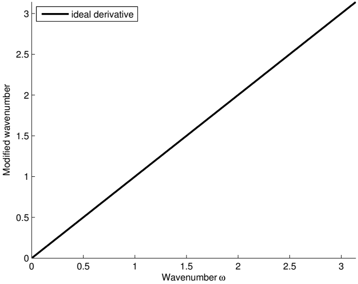

Figure: Ideal Derivative

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Standard Central Difference

Definition (Central Difference Operator)

$$f'(x) \approx \frac{1}{2h}\left[f(x+h) - f(x-h)\right] \tag{3}$$

$$\frac{1}{2h}\begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \tag{4}$$

- frequency response of the central difference operator: $j sin\omega$
- for 2D: rotate the mask for differentiation in y-direction
- with estimates for f_x and f_y, we can compute gradient magnitude and gradient orientation

Introducing Rotation Invariance

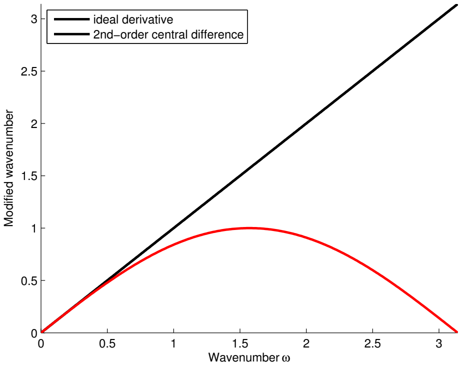

Figure: Central Difference Operator

Sobel, Scharr, Bickley and others

Definition (General 3x3 Differentiation Kernel)

$$D_x = \frac{1}{2h(w+2)} \begin{bmatrix} -1 & 0 & 1 \\ -w & 0 & w \\ -1 & 0 & 1 \end{bmatrix} = \frac{1}{2h} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \frac{1}{w+2} \begin{bmatrix} 1 \\ w \\ 1 \end{bmatrix} \quad (5)$$

- Sobel mask: $w = 2$ (corresponds to smoothing with a binomial kernel)
- Bickley mask: $w = 4$
- Scharr mask: $w = 10/3$ (popular method, that works well in practice)
- NOTE: does not improve gradient magnitude estimation!

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Improving the Central Difference Operator

- frequency response of the central difference operator: $jsin\omega$
- compensate for smoothing effects by applying a binomal kernel orthogonally:
  $\frac{1}{w+2}\begin{bmatrix} 1 \\ w \\ 1 \end{bmatrix}$
- frequency response of the binomial kernel: $S_w(\omega) = \frac{w+2cosw}{w+2}$
- perhaps then an operator with a frequency response of $jsin\omega \cdot \frac{1}{S_w(\omega)}$?

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Padé Schemes

Definition (4th Order Tridiagonal Padé Scheme)

$$\frac{1}{6}[f'(x-h)+4f'(x)+f'(x+h)] \approx \frac{1}{2h}[f(x+h)-f(x-h)] \quad (6)$$

$$\frac{1}{6}(f'_{i-1}+4f'_{\mathrm{i}}+f'_{\mathrm{i}+1}) = \frac{1}{2}(-1\cdot f_{\mathrm{i}\text{-}1}+0\cdot f_{\mathrm{i}}+1\cdot f_{\mathrm{i}+1}) \quad (7)$$

- derived using classical Padé approximations
- lead to tridiagonal systems of linear equations
- frequency response $jsin\omega \cdot \frac{1}{S_4(\omega)}$

Padé Schemes

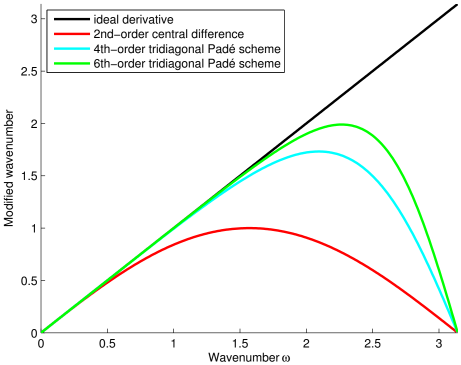

Figure: Padé schemes

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General Implicit Scheme

Definition (General Seven-Point Stencil (Lele, 1992))

$$\beta f'_{i-2} + \alpha f'_{i\text{-}1} + f'_{i} + \alpha f'_{i+1} + \beta f'_{i+2} = c\frac{f_{i\text{-}3} - f_{i\text{-}3}}{6} + b\frac{f_{i+2} - f_{i\text{-}2}}{4} + a\frac{f_{i+1} - f_{i\text{-}1}}{2} \quad (8)$$

$$H(\omega) = \frac{asin\omega + (b/2)sin2\omega + (c/3)sin3\omega}{1 + 2\alpha cos\omega + 2\beta cos2\omega} \quad (9)$$

- set of coefficients using empirical considerations

$$\alpha = 0.5771439, \;\; \beta = 0.0896406,$$
$$a = 1.302566, \;\; b = 0.99355, \;\; c = 0.03750245$$

General Implicit Scheme

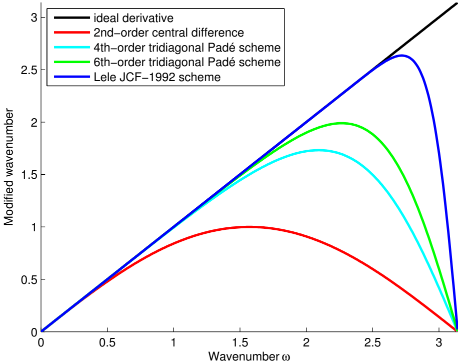

Figure: Lele scheme

Fourier-Padé-Galerkin Approximation

- set of coefficients using a *Fourier-Padé-Galerkin* approximation:

$$\alpha = \frac{3}{5}, \quad \beta = \frac{21}{200}, \quad a = \frac{63}{50}, \quad b = \frac{219}{200}, \quad c = \frac{7}{125}$$

- basic idea:
  - goal: obtain the coefficients of $H(\omega) = \frac{a sin\omega + (b/2) sin2\omega + (c/3) sin3\omega}{1+2\alpha cos\omega + 2\beta cos2\omega}$
  - approximate the ideal derivative $f(\omega) = \omega$
  - use a rational Fourier series: $f(\omega) \approx R_{kl}(\omega) = P_k(\omega)/Q_l(\omega)$

Fourier-Padé-Galerkin Scheme

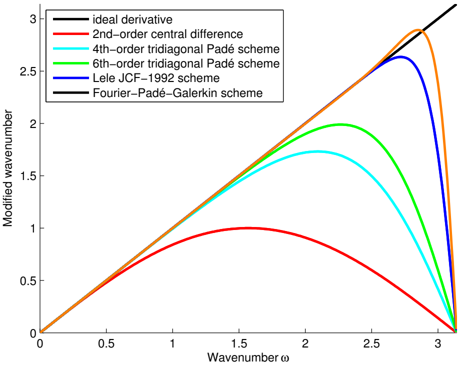

Figure: Fourier-Padé-Galerkin Scheme

Applications

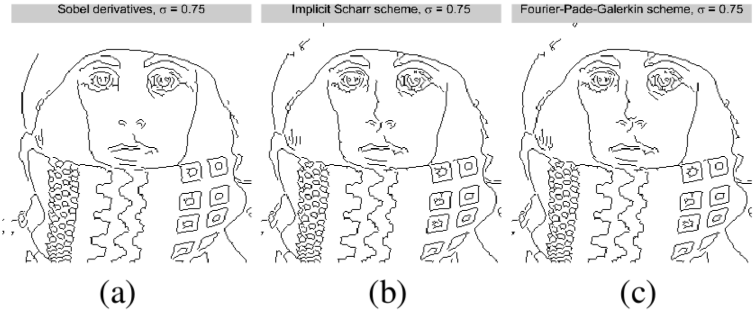

Figure: Canny Edge Detection using various differentiation schemes: (a) Sobel mask, (b) implicit Scharr scheme, (c) Fourier-Pade-Galerkin scheme

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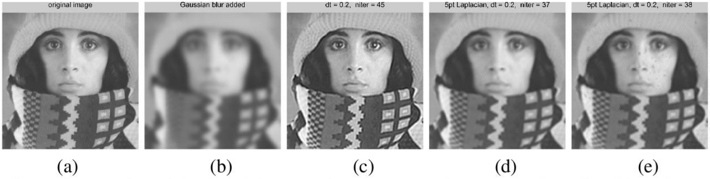

Figure: Deblurring: (a) original image, (b) Gaussian blur applied (c) using an implicit Bickley scheme (d, e) using the explicit Laplacian mask

Summary

- traditional explicit schemes flawed, introduce unwanted smoothing, possible solutions:
  1. apply smoothing in orthogonal direction
     - leads to explicit filter kernels for differentiation (e.g. Sobel, Scharr, etc.)
  2. apply smoothing to derivatives
     - leads to tridiagonal Pade schemes, as well as Fourier-Pade-Galerkin schemes

For Further Reading I

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A.G. Belyaev
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