Differential Geometric Aspects in Image Processing

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Problem C6.1

a) Let $\mathbf{u} = (u, v)$. We have

$$D\sigma(u,v) = \begin{pmatrix} -\sin u \cos v & -\cos u \cos v \\ -\sin u \sin v & \cos u \sin v \\ \cos u & 0 \end{pmatrix},$$

thus

$$\mathbf{I}_{(u,v)} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^2 u \end{pmatrix}.$$

Therefore

$$\begin{aligned} A = \int_{\mathbb{S}^2} dS &= \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{det(\mathbf{I_u})} du dv \\ &= \int_0^{2\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(u) du dv = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(u) du = 2\pi \left[\sin(u)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4\pi \end{aligned}$$

b)

$$(U \circ \sigma)(u,v) = \sin(u)$$

$$\nabla_{\mathbb{S}^2} U = a_1 \frac{\partial \sigma}{\partial u} + a_2 \frac{\partial \sigma}{\partial v}$$

and

$$g^{-1} = \mathbf{I^{-1}_{(u,v)}} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^{-2}(u) \end{pmatrix}.$$

In general

$$a_i = \sum_{j=1}^{N} g^{ij} \partial_j U,$$

thus for our surface

$$a_1 = g^{11}\partial_1 U + g^{12}\partial_2 U = \cos(u)$$

and

$$a_2 = g^{21}\partial_1 U + g^{22}\partial_2 U = 0.$$

Therefore

$$\nabla_\sigma U = \cos(u)\frac{\partial\sigma}{\partial u} = \begin{pmatrix} -\sin u \cos v \cos u \\ \sin u \sin v \cos u \\ \cos^2 u \end{pmatrix}$$

c) We have

$$\sqrt{det(g)} = \cos u$$

and

$$g^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \cos^{-2}(u) \end{pmatrix}.$$

Therefore

$$\Delta_{\mathbb{S}^2} f = \frac{1}{\cos(u)}\frac{\partial}{\partial u}\left(\cos(u)\frac{\partial f}{\partial u}\right) + \frac{1}{\cos^2(u)}\frac{\partial^2 f}{\partial v}$$