Differential Geometric Aspects in Image Processing

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Problem C5.1

i) Fixing θ gives a the radial geodesics in arc-length parametrisation.

ii) Follows from the systems of ode's for a geodesic

$$\rho'' + \Gamma_{11}^1(\rho')^2 + 2\Gamma_{12}^1\rho'\theta' + \Gamma_{22}^1(\theta')^2 = 0 \quad (1)$$

$$\theta'' + \Gamma_{11}^2(\rho')^2 + 2\Gamma_{12}^2\rho'\theta' + \Gamma_{22}^2(\theta')^2 = 0 \quad (2)$$

applied to a radial geodesic ($\theta' = 0$).

iii) From the definition of the Christoffel symbols we have that

$$\sigma_{\rho\rho} = \Gamma_{11}^1\sigma_\rho + \Gamma_{11}^2\sigma_\theta + L_1 N$$

Taking the scalar product with σ_ρ and σ_θ respectively we obtain

$$\Gamma_{11}^1 E + \Gamma_{11}^2 F = \langle\sigma_{\rho\rho}, \sigma_\rho\rangle = \frac{1}{2}E_\rho$$

$$\Gamma_{11}^1 F + \Gamma_{11}^2 G = \langle\sigma_{\rho\rho}, \sigma_\theta\rangle = F_\rho - \frac{1}{2}E_\theta.$$

$\Gamma_{11}^1 = E_\rho = 0$ follows from the first and $F_\rho = 0$ from the second one.

iv) Let $\alpha(t)$ be the geodesic circle through $q \in V \setminus \{p\}$ and let $\gamma(s)$ be the radial geodesic through q in arc-length parametrisation. Then, $F(\rho, \theta) = \left\langle \frac{d\alpha}{dt}, \frac{d\gamma}{ds} \right\rangle$ for all $\rho > 0$.
Fixing $\theta = const$ we get that $\lim_{\rho\to 0} F(\rho, \theta) = \lim_{\rho\to 0} \left\langle \frac{d\alpha}{dt}, \frac{d\gamma}{ds} \right\rangle = 0$

Problem C5.2

i) The tangents of the $2D$ level sets u are given by $(-u_y, u_x)$, since $\overrightarrow{n} = \frac{\nabla u}{|\nabla u|}$. Moreover, since the projections of the level sets of u coincide with geodesic circles, this implies that $\Pi(\overrightarrow{t^{\alpha}}) = c(-u_y, u_x)$

ii) Denoting $\overrightarrow{t^{\alpha}} = (T_1, T_2, T_3)$ we obtain from i) that

$$(T_1, T_2) = c(-u_y, u_x).$$

Furthermore, we have that $N = \frac{(-p,-q,1)}{\sqrt{1+p^2+q^2}}$ and $\left\langle \overrightarrow{t^{\alpha}}, N \right\rangle = 0$. Therefore

$$T_3 = c(qu_x - pu_y).$$

Finally we compute the constant c s.t. $\overrightarrow{t^{\alpha}}$ is normalised. Namely $|\overrightarrow{t^{\alpha}}| = |c|(-u_y, u_x, qu_x - pu_y)| = 1$, thus $c = \sqrt{u_x^2 + u_y^2 + (qu_x - pu_y)^2}$ and the result follows.