Differential Geometric Aspects in Image Processing

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Solutions to exercises: December 5, 2019

Problem C4.1

i) If ϕ is an isometry then in particular choosing $w_1 = w_2 = w \in T_pS$ we have

$$\mathbf{I}_p(w) = \mathbf{I}_p(w,w) = \mathbf{I}_{\phi(p)}(\mathrm{D}\phi(p)(w), \mathrm{D}\phi(p)(w)) = \mathbf{I}_{\phi(p)}(w)$$

Conversely, by linearity notice that

$$\begin{aligned} 2\mathbf{I}(w_1, w_2) &= \mathbf{I}_p(w_1 + w_2) - \mathbf{I}_p(w_1) - \mathbf{I}(w_2) && (1)\\ &+ \mathbf{I}_{\phi(p)}(\mathrm{D}\phi_p(w_1 + w_2)) - \mathbf{I}_{\phi(p)}(\mathrm{D}\phi_p(w_1)) - \mathbf{I}_{\phi(p)}(\mathrm{D}\phi_p(w_2)) && (2)\\ &= 2\mathbf{I}_p(w)(\mathrm{D}\phi_p(w_1), \mathrm{D}\phi_p(w_2)), && (3) \end{aligned}$$

Hence if $\mathbf{I}$ is preserved, ϕ is an isometry

ii) Let $\tilde{\sigma} : U \to S$ by the parametrization of the cylinder

$$\tilde{\sigma} = (\cos u, \sin u, v)$$

and $\sigma : U \to P$ be given by

$$\sigma(u, v) = (u, v, 0)$$

Consider $\phi = \sigma \circ \tilde{\sigma}^{-1}$ We show that ϕ is a local isometry between S and $\{z = 0\} \subset \mathbb{R}^3$. In fact If $w \in T_pS$ for some $p \in \tilde{\sigma}(U)$ then

$$w = \tilde{\sigma}_u u' + \tilde{\sigma}_v v'.$$

Moreover, $\mathrm{D}\phi(p)(w)$ is tangent to the curve $\phi(\tilde{\sigma}(u(t), v(t))) = \sigma(u(t), v(t))$, hence

$$\mathrm{D}\phi(p)(w) = \sigma_u u' + \sigma_v v'$$

Moreover, since $E = \tilde{E} = 1, F = \tilde{F} = 0, G = \tilde{G} = 1$, we obtain

$$\begin{aligned} \mathbf{I}_p(w) &= \tilde{E}(u')^2 + 2\tilde{F}u'v' + \tilde{G}(v')^2 && (4)\\ &= E(u')^2 + 2Fu'v' + G(v')^2 = \mathbf{I}_{\phi(p)}(\mathrm{D}\phi(p)(w)) && (5) \end{aligned}$$

iii) Consider the parametrisation $\sigma(u,v) = (cosu, sinu, v)$ around a point $p \in S$ with $\sigma(0,0) = p$. In a neighborhood of p an arc length parametrised curved α can be expressed as $\sigma(u(s), v(s))$. Moreover, from ii), σ is a local isometry. Therefore the curve $\alpha(s)$ is a geodesic if and only if $(u(s), v(s))$ is. But the geodesics of a plane are straight lines and the only option is then

$$u(s) = as, \quad v(s) = bs, \quad a^2 + b^2 = 1$$

Going back to the cylinder, we obtain that the geodesic are of the form

$$(\cos as, \sin as, bs)$$

Problem C4.2

Replacing the expressions for the Christoffel symbols, the system of differential equations satisfied locally by curves with parallel derivative (Lecture 12, Slide 10) becomes

$$u'' + \frac{2ff'}{f^2}u'v' = 0 \tag{6}$$

$$v'' - \frac{ff'}{(f')^2 + (g')^2}(u')^2 + \frac{f'f'' + g'g''}{(f')^2 + (g')^2}(v')^2 = 0 \tag{7}$$

For the meridians $u = const$ and equation (6) is clearly satisfied.
On the other hand, the first fundamental form for a surface of revolution gives $\langle \sigma_v, \sigma_v \rangle = (f')^2 + (g')^2$. For the meridian parametrised in arch length, $\sigma(const, v(s))$, this leads to

$$\left((f')^2 + (g')^2\right)(v')^2 = 1$$

or

$$(v')^2 = \frac{1}{((f')^2 + (g')^2)}. \tag{8}$$

and deriving the last equation w.r.t. s

$$2v'v'' = -\frac{2(f'f'' + g'g'')}{((f')^2 + (g')^2)^2}(v').$$

Finally applying (8) to the last equation we get

$$v'' = -\frac{2(f'f'' + g'g'')}{(f')^2 + (g')^2}(v')^2,$$

which corresponds to (7) for u constant.
The curve $\sigma(u, v(s))$ therefore has a parallel tangent vector and is arc-length parametrised: It is a geodesic.