Differential Geometric Aspects in Image Processing

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Problem C4.1

Recall: A diffeomorphism $\phi : S \to \bar{S}$ is an **isometry** if for all $p \in S$ and all pairs $w_1, w_2 \in T_p(S)$ we have that

$$\mathbf{I}_p(w_1, w_2) = \mathbf{I}_{\phi(p)}(\mathrm{D}\phi(p)(w_1), \mathrm{D}\phi(p)(w2)).$$

i) Denote with $\mathbf{I}_p(w) := \mathbf{I}_p(w, w)$, the norm induced by the first fundamental form, for all $w \in T_pS$. Show that a diffeomorphism ϕ is an isometry if and only if

$$\mathbf{I}_p(w) = \mathbf{I}_{\phi(p)}(\mathrm{D}\phi(p)(w))$$

for all $w \in T_pS$

ii) Show that the cylinder $\left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1\right\}$ is locally isometric to the plane $\{(x, y, z) : z = 0\} \subset \mathbb{R}^3$.

iii) Compute the geodesics of the cylinder.

Problem C4.2

i) Let $f > 0, g$ be real valued functions with $f > 0$ and let S be a surface of revolution (rotation surface) parametrised by

$$\sigma(u, v) = (f(v) \cos u, f(v) \sin u, g(v)).$$

i) Show that the meridians of S are geodesics

Hint: Its Christoffel symbols are given by

$$\begin{aligned} \Gamma^1_{11} = 0, \quad \Gamma^2_{11} &= -\frac{ff'}{(f')^2 + (g')^2} \\ \Gamma^1_{12} = \frac{ff'}{f^2}, \quad \Gamma^2_{12} &= 0 \\ \Gamma^1_{22} = 0, \quad \Gamma^2_{22} &= \frac{f'f'' + g'g''}{(f')^2 + (g')^2} \end{aligned}$$