Differential Geometric Aspects in Image Processing

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Problem C3.1

Draw a segment of prescribed length L orthogonal to the tangent line from each point of c. The endpoints of this family of segments constitute the graph of the the level set at distance L.

Problem C3.2

Let $\phi : U \to \phi(U)$ be a local parametrisation of the surface M s.t. U is a path connected. We already showed in the previous lecture that if all points are umbilic then $\kappa = \kappa_1 = \kappa_2$ is constant. There are two options:

i) If $\kappa = 0$, then clearly the surface M is contained in a plane.

ii) If $k \neq 0$ then define the map $Y : U \to \mathbb{R}^3$, $Y(u,v) = \phi(u,v) + \frac{1}{\kappa} N(u,v)$ with N the Gauus map. Then

$$Y_u = \phi_u + \frac{1}{\kappa}\mathrm{D}N\phi_u = \phi_u - \frac{1}{\kappa}\kappa\phi_u = 0 \tag{1}$$

$$Y_v = \phi_v + \frac{1}{\kappa}\mathrm{D}N\phi_v = \phi_v - \frac{1}{\kappa}\kappa\phi_v = 0. \tag{2}$$

Thus Y is constant and $|\phi - Y|^2 = \frac{1}{|\kappa|}$. This means that $\phi(U)$ is in a sphere with center Y and radius $\frac{1}{\kappa}$. The result follows from the path connectivity of M.

Problem C3.3

First of all, notice that $\sigma(u,v) = A \cdot (u, v, f(u,v))^\top$ for the orthogonal matrix

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

We call $\tilde{\sigma} = A \circ \sigma$.

i) $\tilde{\sigma}$ is a parametrisation for the surface and

$$\tilde{\sigma}_u \times \tilde{\sigma}_v = (1, 0, f_u)^\top \times (0, 1, f_v)^\top = (-f_u, -f_v, 1)^\top \neq (0, 0, 0)^\top$$

ii) $n = \frac{\tilde{\sigma}_u \times \tilde{\sigma}_v}{||\tilde{\sigma}_u \times \tilde{\sigma}_v||} = \frac{1}{\sqrt{1+|\nabla f|^2}}(-f_u, -f_v, 1)^\top$

iii) Let $\boldsymbol{u} = (u, v)$. The first fundamental form is

$$\mathbf{I}_{\boldsymbol{u}} = \begin{bmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{bmatrix}$$

Notice that $\tilde{\sigma}_u \perp n$, which implies

$$\partial_u(< n, \tilde{\sigma}_u >) =< n_u, \tilde{\sigma}_u > + < n, \tilde{\sigma}_{uu} >= 0,$$

hence

$$< \tilde{\sigma}_u, n_u >= - < \tilde{\sigma}_{uu}, n > .$$

From similar considerations we obtain that

$$\mathbf{II}_{\boldsymbol{u}} = \begin{bmatrix} n\tilde{\sigma}_{uu} & n\tilde{\sigma}_{uv} \\ n\tilde{\sigma}_{uv} & n\tilde{\sigma}_{vv} \end{bmatrix}$$

thus

$$\mathbf{II}_{\boldsymbol{u}} = \frac{1}{\sqrt{1 + |\nabla f|2}} \begin{bmatrix} f_{uu} & f_{uv} \\ f_{uv} & f_{vv} \end{bmatrix}$$

iv) Using the fact that $S_p = \mathbf{I}_{\boldsymbol{u}}^{-1} \mathbf{II}_{\boldsymbol{u}}$ and

$$\mathbf{K} = \det S_p \quad \mathbf{H} = \operatorname{tr}(S_p)$$

the result follows.